

REPORT No. 28.

AN INTRODUCTION TO THE LAWS OF AIR RESISTANCE OF AEROFOILS.

By GEORGE DE BOTHEZAT.

PREAMBLE.

It is only very slowly, through the centuries, that the notion of the resistance of a fluid to the motion of a solid body has been developed. This notion is intimately associated with the concepts which we gain from mechanical phenomena. In the aurora of the first gleams which pierced the darkness of the human mind in the domain of the concepts of motion, fluid resistance was not differentiated from motion. Thus Aristotle¹ considered in principle—not willing to admit the possibility of a vacuum—that the resistance of a fluid was inseparable from the phenomena of motion. It is this point of view which paralyzes, so to say, completely his attempts to form a conception of the phenomena of motion, the exposition of which by him was, it must be added, very hazy. Through antiquity to the Middle Ages, dynamical phenomena were dawning, but with a very confused misunderstanding. Leonardo de Vinci seems to have thought much about the motion of bodies under terrestrial conditions. It is without any doubt that he made numerous and remarkable attempts at mechanical flight. But, in his dynamical concepts, he does not seem to have clearly separated the phenomena of motion from the phenomena of the resistance of fluids. Thus he used the confused conception of the *impetus* which ought to be communicated to a body when the same is set in motion, and which ought to dissipate itself progressively to cause the body to stop. But by the use of the conception of dissipation of the impetus, he even arrived at the happy conclusion of the impossibility of *perpetuum mobile*. It is Galileo² who finally has a full conception of the material nature of the gases and of the influence the same have on the motion of bodies—an influence which he knew to decrease with the velocity. This is why Galileo, in his celebrated experiments on falling bodies, recognized the necessity of making them at low velocities. Low velocities first made possible the quantitative observations, and secondly diminished all the resistance, for the decreasing of which all possible measures were taken. So it is that Galileo first came to the modern conception of dynamical phenomena.

To disengage the law of the motion of bodies, considering the latter as moving *in vacuo* and without any kind of resistance, and to look on all other effects, such as friction or medium-resistance, as additional effects, this was the conception which allowed the establishment of dynamics. This conceptional sorting of questions in the complex problem of motion must be considered as one of the greatest scientific conquests. On our planet the motion of bodies always takes place in a fluid. The phenomena of motion, taken as a whole, is so complex that it is inextricable for the human mind. A very large conceptional effort had to be developed to rise to the abstraction of the phenomena of motion, cleared from the influence of the immediate medium. But once this big step made, we have the magnificent picture of the powerful dynamical laws, of which we have seen the development; the questions of friction and fluid resistance being considered as special separate questions, whose complexity is enough to make them subjects

¹ Aristotle. "Physique," Livre VII, Chap. VI. French translation by Barthelmy Saint-Hilaire.

² Galileo. "Discorsi et dimostrazioni matematiche intorno a due nuove scienze attenanti alla meccanica et i movimenti locali." Bologna edition 1655.

of special branches of science. We understand now why all the first dynamical investigations, until recent times, were almost exclusively made on problems wherein the influence of the medium could be neglected. It is for this last reason that the advance of celestial mechanics was developed long before we began to understand the laws of the motion of bodies under terrestrial conditions. It is undoubtedly true that it is only with the birth of aviation that profound studies of the motion of bodies in *real fluids* were started and that the light began to penetrate through the complex and delicate phenomena of fluid resistance, which phenomena have, for a long time, veiled from our eyes the laws of dynamics but which have now given us the conquest of the aerial ocean.

Actually we are only taking the first steps in the conception of the problems of fluid resistance. The former status of these questions consisted more in the comparison of fluids to some mechanical system, more or less similar to fluids, than in the study of the real fluids with their real properties. Thus, Newton likened fluids to a system of elastic particles whose impact on the solid body produced the fluid resistance. Euler¹ in his research on fluid resistance, likened fluids to a continuous homogeneous frictionless medium and calculated the fluid resistance by aid of the general equations which he built up for that kind of medium. He was brought to the conclusions, very far from reality, that a body moving in a fluid meets no resistance to its motion. This conclusion is a consequence of the assumption of a continuous and noncyclic flow around the body. Recently Kutta has shown that, in the general case of the continuous flow of a perfect fluid around the body, the circulation around the contour embracing the body can have a finite value, and in such a case the fluid resistance has a finite value but is perpendicular to the general stream velocity. Thus, in a perfect fluid only the power corresponding to the resultant pressure on the surface of the body is necessarily equal to zero, but the resultant pressure can have a finite value. We will later consider Kutta's conceptions. Helmholtz is the first to have made a serious attempt to bring the foundation of hydrodynamics into more close agreement with reality; and his work in that sense is of great importance. He showed the necessity for the consideration of vortex motion and indicated the possibility of the formation of surfaces of discontinuity in fluid motion.

This last idea of surfaces of discontinuity was used by Kirchhoff² and by Lord Rayleigh³ for the calculation of fluid resistance in some simple cases, which method was recently largely developed by G. Greenhill⁴ H. Levy⁵ and others. The flow which in reality is established seems only rarely to be of the kind assumed by Kirchhoff and Lord Rayleigh, so that in general the experimentally measured fluid resistance does not correspond to that calculated by the Kirchhoff and Lord Rayleigh method, a fact to which already William Thomson⁶ (Lord Kelvin) has drawn attention. The way in which viscosity has been considered until now does not give a satisfactory solution of the problem of fluid resistance, either. The calculation of fluid resistance by the equations of motion of a viscous fluid in the final form given them by Stokes⁷ seems to agree with experiment only for very small velocities. It is only the development of aviation that has given a new powerful impulse to aerodynamics, and has brought with it the necessity of a conception of fluid resistance closer to reality. Many quite new ideas and concepts have thus been progressively developed.

In 1902 W. M. Kutta⁸ formulated, first for a particular case and soon after generalized for the general case, an important theorem which gives the relation between the fluid resistance and the flow around a body which encounters that resistance. This theorem was established by its author for the case of perfect fluids. In that case, this theorem tells us that the lift of

¹ Euler. "Principes généraux du mouvement des fluides." Histoire de l'Acad. de Berlin 1755. "De principis motus Fluidorum." Novi Comm. Acad. Petrograd 1759.

² Kirchhoff. "Vorlesungen über Mechanik," 1897. 22st Vorlesung.

³ Lord Rayleigh. "Scientific papers," I.

⁴ G. Greenhill. "Stream lines past a plane barrier, and of the discontinuity arising at an edge." Report 19, Advisory Committee for Aeronautics, 1912.

⁵ H. Levy. "Discontinuous motion past a Curved Boundary," Proc. Roy. Soc. 1916.

⁶ Lord Kelvin. "Math. and Phys. papers," IV, p. 215.

⁷ Stokes, G. G. "Math. and Phys. papers," I, p. 78.

⁸ W. M. Kutta. "Illustrirte Aeronautische Mittheilungen," 1902. "Sitzungsberichte der Königl. Bayerischen Akademie der Wissenschaften." München, 1910 and 1911.

the body is equal to the product of the density, velocity of the stream at infinity, and circulation around a contour surrounding the body, but for the drag it gives the value zero. To calculate the lift by this theorem, it is necessary to know the flow around the body. Kutta assumes that this flow is a compound forward and cyclic motion of the fluid; but this is only an assumption, without sufficient foundation. The Kutta theorem, understood as just stated, was applied with many developments to numerous cases by Joukowski¹ and Tchapligne. The results of all these calculations do not fully agree with experiment. I have submitted the Kutta theorem to critical examination² and have showed that this theorem must not be understood as giving the solution of the problem of fluid resistance, because it leaves open the question of the flow around the body and only gives the relation between flow and fluid resistance.

In recent years, Karman³ has called attention to the fact that the flow around a body having a rectilinear and uniform motion of translation in a fluid very often consists of a system of vortices which are formed behind the body, and has shown the relation which must exist between the momentum of these vortices and the fluid resistance.⁴

The works of Karman are probably the first to indicate the necessity of the determination of the type of flow which in reality takes place around a body in order to be able to calculate its fluid resistance, a question to which not enough attention was paid before. And what is particularly important, the type of flow which most generally establishes itself is not necessarily one of the types which were presupposed by all the foregoing theories. The flow around a body immersed in a fluid is not necessarily continuous as it was supposed by Euler; it is not generally characterized by a system of surfaces of discontinuity either, as was assumed by Kirchhoff and Lord Rayleigh, which surfaces of discontinuity must be considered as almost unstable, the viscosity disturbing them; but more often the flow is characterized by a system of vortices as shown by Karman.

Nevertheless, the systematical study of the different kinds of flow around solid bodies which are compatible with the general equation of hydrodynamics is of the highest value. It is of the greatest importance to disengage all the types of flow which are possible for fluids because only under such conditions can we reach the complete solution of the great fluid resistance problem. Generally speaking, all kinds of flow satisfying the equations of hydrodynamics are virtually possible under special conditions. Particular attention must, however, be paid to the question of finding out the exact conditions under which each kind of flow can take place. In many cases of flow of air or water the types of flow characterized by vortices in quincunx seem to be most usually obtained. This is on account of the need of stability, and the conditions of energy dissipation inside those fluids. If we look over the historical development of hydrodynamics it is the progressive discovery of the properties of the different types of flow that we see before us.

In the development of modern hydrodynamics the question of the conditions which fix the type of flow established under given conditions was left nearly without any examination. Exactly speaking, what did the classical hydrodynamics give us in order to determine the flow in the case of steady motion? Of the four equations of the motion of an incompressible fluid which forms the foundation of classical hydrodynamics, three give the relation between the distribution of the velocities and the pressures⁵—it is these which express the theorem of momentum in its application to a fluid particle—and only one, the equation of continuity determines the flow. The question of finding the flow around a body as defined by the equation of continuity is a problem of finding a function which verifies the Laplace equation and satisfies the boundary conditions. It must be remembered, however, that the equation of continuity is only a necessary condition for continuity and is not at all sufficient. As Helmholtz has first remarked, the discontinuity of the tangential components of the velocity in

¹ See "Aerodynamique" by N. Joukowski. Paris, 1916.

² See Note I at the end of this pamphlet.

³ v. Karman. "Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen." 1911. "Physikalische Zeitschrift." 1912. See also the above-mentioned Aerodynamique, by N. Joukowski and Note IV at the end of this pamphlet.

⁴ In the following the conceptions of Karman will be extended to the aerofoil.

⁵ This is particularly well seen when we use the equations of fluid motion in natural curvilinear coordinates. See Note II.

regard to some surfaces is compatible with the equation of hydrodynamics, so that when a flow, satisfying the equation of continuity is found, it must still be verified that such a flow is virtually possible. This is probably one of the most important questions in the problem of fluid resistance. I must finally add that in some cases the continuous flow of a fluid seems to be practically impossible.

I will give an example. To follow more easily the motion of a fluid, let us divide its continuous volume by a system of triorthogonal surfaces which accompany the fluid in its motion, so that we have no flow through these surfaces. Continuous motion will mean that each fluid element will always remain in contact with the 14 elements which are in touch with it at any moment; that all the elements contained in any closed surface moving with the fluid will always remain in it; that all the elements which are inside the fluid will never come on its surface; that all the elements which are on the boundary surface of the fluid will never come inside the fluid, etc.; so that the whole motion is considered only as a continuous deformation of the fluid medium without any alteration of the mutual grouping of the elements. If we now consider for example the flow of a viscous fluid running out of a pipe into a reservoir, considering, as generally admitted, the velocity of the fluid on the pipe walls equal to zero, and if we attempt to follow the deformation of a fluid element, we very easily see the impossibility of such a conception. It is enough to remember that the elements all keeping close together will be found in some cases making some hundreds of thousands of revolutions per second.¹ The admittance of continuity in such conditions seems to be very difficult. In all probability, the real motion must consist of a succession of continuous states of motion interrupted by discontinuous intervals.

The following question can very naturally arise: How did it happen that in the domain of rigid dynamics we at once reached so many results which stay in close agreement with the motion of real solid bodies, and that in many hydrodynamical problems we have not been till now able to secure satisfactory solutions. The fact lies in the nature of the question. In the historical evolution of mechanics the concept of a rigid body was first fully reached. The formation of this concept did not present any special difficulties and its application to the analysis of an enormous number of problems of practical mechanics has shown at once all its power. The scientific world was already in the possession of a fully developed rigid dynamics, experimentally verified, when, in Euler's times, attention was brought to the general problem of fluid motion. When the concept of a perfect fluid was reached it was instinctively assumed that this conception bore a relation to the real fluid quite as close as the conception of rigid bodies to a real solid body. It was with great astonishment that men recognized the disagreement which began to appear between the consequences of the hydrodynamical equations and the hydraulic experiments. For a long time the investigators in hydrodynamics somewhat skeptically considered the disagreement between theory and practice, and did not pay much

¹ In the case of a continuous motion of a fluid in a horizontal pipe, the axis of the pipe is an axis of symmetry for the whole phenomenon. Using this axis, as the Z axis of a system of cylindrical coordinates, the equations of the fluid motion in these coordinates for our case will be

$$\delta \mu \frac{\partial v_z}{\partial z} = \mu \frac{\partial^2 v_z}{\partial r^2} - \frac{\partial p}{\partial z}$$

Where δ is the density of the fluid and μ is the viscosity constant. But in consequence of continuity and incompressibility

$$\frac{\partial v_z}{\partial z} = 0 \text{ and } \frac{\partial p}{\partial z} = A = \text{Const.}$$

where A is the pressure gradient along the Z axis, so that

$$\mu \frac{\partial^2 v_z}{\partial r^2} = A$$

and

$$\mu v_z = \frac{1}{2} A r^2 + B$$

For the vortex components we have

$$\omega_r = 0; \omega_z = 0; \omega_\theta = \frac{\partial v_z}{\partial r}$$

and from the foregoing

$$\omega_\theta = \frac{A r}{\mu}$$

In some of Poiseuille's experiments with water, A was of the order of one atmosphere $\approx 10^6 \frac{\text{dynes}}{\text{cm}^2}$; $\mu \approx 0.01$; $r \approx 0.001$

so that we get

$$\omega_\theta \approx 10^4;$$

which gives $10^4/2\pi$ revolutions per second.

attention to it; and I will allow myself to say that probably some of them believed more the conclusions of their equations rather than the experimental results. The great success of rigid dynamics in its origin is without any doubt one of the principal reasons why there has been so much confidence in the concept of a perfect fluid and why in a certain period of the development of hydrodynamics this science has been brought to a very abstract development, more as a mathematical discipline than as a science of nature. But the demands of the magnificent conquest of the aerial ocean by the airplane has, I think, definitely brought the hydrodynamical science on the right way of one of the most important natural sciences.

If we review the foregoing, we can now give the following statement of the question of fluid resistance:

To be able to calculate the fluid resistance of a body, we must first determine the type of flow which takes place around the body in the case considered.

It appears that the conditions which hold at the surface of contact of fluid and solid constitute a special difficulty and hence that special conditions exist there. It may therefore seem that it is necessary first to make a special study of the problem of the flow of the fluid in the immediate neighborhood of the body. As conditions of flow depend upon the shape of the surface of the body and the physical properties of the fluid, I think that the solution of this problem could be obtained only in an empirical-theoretical way; that is to say, to find out by what quantities, experimentally measured, we can fix the mutual relation between the surfaces of contact and the fluid flowing along them, so that these quantities once known, the flow in the neighborhood of the body could be determined. It seems that only a thin layer of fluid is disturbed by the immediate influence of the surface of a body and that at a moderate distance from the body the influence of the body surface practically disappears.

The conditions of flow in the portion of a fluid remote from any rigid body seem to be easier to understand than the conditions in the immediate neighborhood of a body. In the remote fluid portions we can have continuous motion, and so long as continuous motion takes place *no vortices can appear within the fluid, and this independently of any assumption as to viscosity.*¹ The appearances of vortices can only come from the formation of *surfaces of discontinuity* in the fluid. The mechanics of formation of the latter surfaces is very probably the following:

A real fluid has to be considered as a fluid-elastic body (in opposition to the solid-elastic body), the stresses in which are fixed by the distribution of the velocity gradient. The fluid-elastic body can, without any doubt, move as a continuous whole only provided the stresses at all the points of the fluid have not reached a certain value. If some of these stresses exceed a certain magnitude, which must depend upon the properties of the fluid, the fluid may break at that point if tension stresses appear, or slip, if the stresses are shears. It is in this way that surfaces of discontinuity arise in a fluid. But the existence of them can be only a momentary phenomenon which is replaced by vortices, the surfaces of discontinuity being unstable in regard to viscosity. We thus see that the study of the problem of fluid resistance must consist first, of finding out the conditions under which continuous motion of a fluid can take place around a body. The system of stresses in the fluid around the body seems to be the criterion for that continuity. When the latter conditions are not satisfied, then we shall have to find out what systems of vortices can be compatible with the problem; then, afterwards, when the type of flow is exactly fixed, the fluid resistance can be calculated by the theorem of momentum.

¹ Let us consider, for simplicity, a viscous fluid moving parallel to a plane. (For the notations, see Note II.) The moment of the forces acting on two opposite sides of a fluid element will be

$$\frac{\partial T_r}{\partial y} dx dy dz \frac{dy}{2}$$

that is, an infinitely small quantity of the fourth order.

The moments of momentum of the element will be proportional to

$$2 dx dy dz (dy^2 + dz^2),$$

that is, an infinitely small quantity of the fifth order. Therefore, if the forces acting on the surface of the element give rise to a moment, the element will necessarily take an infinite rotation, which would mean discontinuity.

I will here ask the reader to pay special attention to the following fact. The formation of special types of vortex systems behind a body moving in a real fluid is a direct consequence of the energy dissipation inside the fluid owing to viscosity. As will be shown in this pamphlet, the work of the fluid resistance forces brings with it the necessity of the formation behind the body, in the limiting case, not only of the interior surfaces of discontinuity of the Kirchhoff-Lord Rayleigh theory, but also of a system of exterior surfaces of discontinuity remote from the body. All these surfaces of discontinuity are constituted by vortex sheets. But such vortex surfaces of discontinuity being unstable, they go over into stable vortex systems, the quincunx vortex system being the one most generally obtained, for the case of large aspect ratio. I call *fundamental wave* the vortex motion generated by the exterior vortex surface of discontinuity, and *secondary wave* the vortex motion generated by the interior vortex surface of discontinuity. We can now understand why at small velocities the flow around a body approaches more a continuous flow. At small velocities the work of the fluid resistance forces is small and is quickly dissipated inside the fluid. But at greater flow velocities the work of the fluid resistance forces can not be at once dissipated in the fluid, and a decrease of the kinetical and potential energy of the fluid is produced, which gives rise to an oscillatory motion of the fluid left behind the body, and thus a progressive dissipation of the lost energy is realized.

We thus see that the whole question of the problem of air resistance consists in finding out the conditions which determine the kind of flow around a body, and we see now how far the first attempts to calculate the fluid resistance were from reality. They can only be considered as attempts to draw the conclusions from certain assumptions, and it is only with time that the idea of the conception of a real fluid, which was always problematic, has slowly been reached; and we find ourselves now only at the beginning of the development of this great question.

This pamphlet must be considered only as an introduction to the question of the law of air resistance of aerofoils, which will give a general review of the present main knowledge of that question. But a special attempt will be made to show the insufficiency of many conceptions often admitted, and to indicate the ways in which, it seems to me, future investigations must be undertaken. At the end I have added some notes which I think will be of interest for those who would like to have more complete references concerning the questions discussed.

Among the questions contained in this pamphlet the following are taken from the author's lectures, given since 1912, at the Polytechnical Institute of Petrograd: The scheme of the phenomenon of fluid resistance; calculation of the apparent angle of deflection of the stream behind an aerofoil; the establishment of the fundamental wave created by the motion of an aerofoil and the determination of its characteristic elements; determination of the part of the drag due to tip vortices and its dependence upon aspect ratio; connection between tip and edge vortices and the relation of the last to the drag and the lift of the aerofoil; generalization of Bernouilli's theorem; exact demonstration and generalization of Kutta's theorem; the equation of metacentric curves in Plucker's coordinates.

The author takes pleasure in thanking Dr. J. S. Ames for his kind assistance given by reading the manuscript of this Report and correction of its style.

GEORGE DE BOTHEZAT.

WASHINGTON, D. C., *September, 1918.*

CHAPTER I.

PRELIMINARY CONSIDERATIONS.¹

When, under earth conditions, a certain body is moving, its motion necessarily takes place in a fluid, more generally in air or water.

When the velocity of the body is relatively low, or the fluid is of low density and low viscosity, the action of the fluid medium on the motion of the solid body is not very marked. In those cases we can, without appreciable error, abstract ourselves from the influence of the medium and consider the motion of the body as taking place in a vacuum.

When the velocity of the body reaches a certain value in a viscous fluid of finite density, the action of the medium on the motion of the body becomes of prime importance. In that case, to be able to study the motion of a solid body, we must, in addition to the forces which act on the body and among which we necessarily have the Archimedes lift, add a system of forces which express the action of the fluid on the different elements of the surface of the body in motion. This system of superficial forces, *which is distributed over all the surface of the body*, is generally called *fluid resistance*.

For any body having any general motion in a fluid, the determination of the fluid resistance is so complex a problem that its general solution can actually not be found either experimentally or theoretically. Only some very simple cases of uniform and rectilinear motion of bodies have been, until now, submitted to a more or less complete investigation.



FIG. 1.

We imagine a solid, which is brought into motion with a rectilinear and uniform velocity of translation, in a fluid medium, which is immobile with respect to the earth, which has uniform and constant temperature, and which has such dimensions that the disturbances caused by the motion of the solid do not reach the boundary surface of the fluid. In that condition, at a time which is generally somewhat after the body has reached this constant velocity, certain steady conditions are established. The solid is, so to say, accompanied in its motion through the fluid by a certain state of disturbance of the fluid around it. There was a time when it was thought that this disturbance has, relative to the body, an invariable configuration; but we now know that, generally, this disturbance is invariable relative to the body only *before* the same, and that *behind* it we often have a state of periodical disturbance. The result of this disturbance is a system of steady or periodical forces acting on the whole surface of the solid. This system of forces, which constitutes the fluid resistance, can always be brought to a resultant wrench, whose components will be designated by \bar{R} , for the resultant force of the wrench, and by \bar{C} , for the resultant torque of the wrench. (See fig. 1.) If these above-mentioned forces are periodical, we will understand by \bar{R} and \bar{C} the mean values of the resultant force and the resultant torque of the wrench.

It is quite possible that, for the same body brought into motion with the same velocity, the system of forces of air resistance may be different, depending on the manner in which the body is brought to its state of motion, but it seems that in most general cases, the viscosity tends, so to say, to make uniform all the possible types of disturbances around the body, so that, generally, in a free fluid, the same disturbances are always established around the body when it reaches the same velocity in the same fluid. In that sense we can say:

For a solid body, moving in a fluid medium with a constant velocity, there corresponds a determinate fluid resistance.

¹ The main contents of this chapter are taken from the first chapter of the Author's "Étude de la Stabilité de l'Aéroplane" Paris 1911, a chapter which was written at that time under the influence of the lectures of Paul Painlevé.

We therefore see that, when the body has reached a steady state of motion, the resultant force \bar{R} and the resultant torque \bar{O} of the wrench of the fluid resistance are independent of the time, and are functions only of the magnitude of the velocity and its orientation toward the body.

Under the "law of fluid resistance for uniform translation of a solid," we will understand the formulas which give, in position and magnitude, the resultant force \bar{R} and the resultant torque \bar{O} of the wrench of fluid resistance, as functions of the characteristics of the form and the dimensions of the body under consideration, and the magnitude of the velocity of the body relative to the fluid and its orientation toward the body.

It must be remarked that the components \bar{R} and \bar{O} of the resultant wrench do not replace fully the fluid resistance, but are equivalent to the system of forces of fluid resistance in only one single sense, namely, that modification which the fluid resistance introduces in the *motion* of the body will be the same when we replace the system of forces of fluid resistance by the resultant wrench. In all other relations \bar{R} and \bar{O} are not equivalent to the fluid resistance; for example, the stresses which are produced in the body by the system of fluid resistance are entirely modified when we substitute \bar{R} and \bar{O} for the fluid resistance. The resultant wrench of a system of forces is only an analytical transformation, the possibility of which is established by the theorems of mechanics and which allows us to reduce a given system of forces to its simplest expression. The resultant wrench is determined only with one degree of freedom, its position on its line of direction being entirely arbitrary. There is no interest in seeking for an exact position of the wrench on its line of direction. These data would not give us any complementary indication upon the motion of a solid. The motion of a solid is absolutely determined when the resulting wrench of the system of acting forces is given in magnitude, direction, and sense. The position of the wrench on its direction does not enter into the question of motion.

It is easy to see that the system of forces of fluid resistance can never be reduced to a single resultant torque, because, if that were possible, the body once brought to that velocity at which this could take place would be able to move of itself infinitely forward without any expense of power, because it would only be necessary to equilibrate by an acting torque the torque of fluid resistance; and this is in full contradiction to all we know about fluid resistance.

For the same reason the projection of the resultant force \bar{R} on the direction of the velocity must always have a sense inverse to that of the velocity, because if it were not so, the body once brought to that state of motion at which that could happen would be able, for example, to pull something infinitely—the torque of fluid resistance being equilibrated by an acting torque—and so do work of itself, which would be in contradiction with the principle of energy.

When the body under consideration has a plane of symmetry parallel to its velocity, the resistance of the fluid is reduced to a unique resultant force \bar{R} lying in the plane of symmetry of the solid and whose projection on the direction of the velocity has always the inverse sense of the velocity.

This proposition can be easily justified. The system of forces of resistance will then be a symmetrical system (see fig. 2) and can always be reduced to a system of forces lying in the plane of symmetry; but the latter system of forces can always be reduced to a resultant force or to a resultant torque. As we have seen, however, the reduction to a torque being impossible, the system of forces will reduce itself to a single resultant force, the projection of which on the velocity must have a sense inverse to the velocity for reasons already indicated.

All the foregoing does not exclude the possibility of the body's taking a rotary motion as a result of the translatory motion in a fluid.

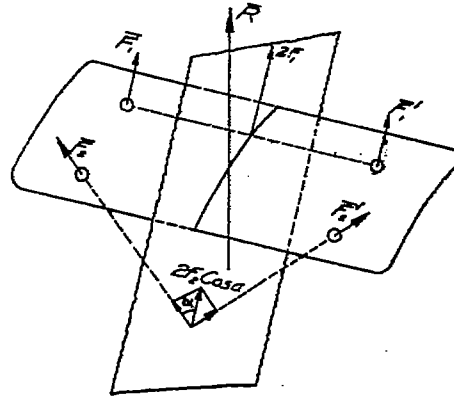


FIG. 2.

We will have especially to study the law of air resistance of aerofoils. The model aerofoil generally has a perimeter of approximately rectangular shape and its cross section is built up by a system of arcs. In figure 3 is represented in plan and in cross section through its plane of symmetry, an aerofoil of the type mentioned, which is moving in air with a constant velocity V parallel to its plane of symmetry. The force of air resistance admits necessarily a resultant force \bar{R} , whose projection on the direction of the velocity is in the inverse sense of the velocity.

I once more emphasize the fact that no point of the resultant force \bar{R} has to be distinguished from any other point and that the resultant force is fully specified when we know, first, its magnitude; second, its line of action, which is defined by direction and position toward the aerofoil under consideration. The position of the resultant force on its line of action is *ad libitum*; that is to say, no special point can be, from a mechanical standpoint, exclusively distinguished as point of application or center of pressure of the resultant force \bar{R} .

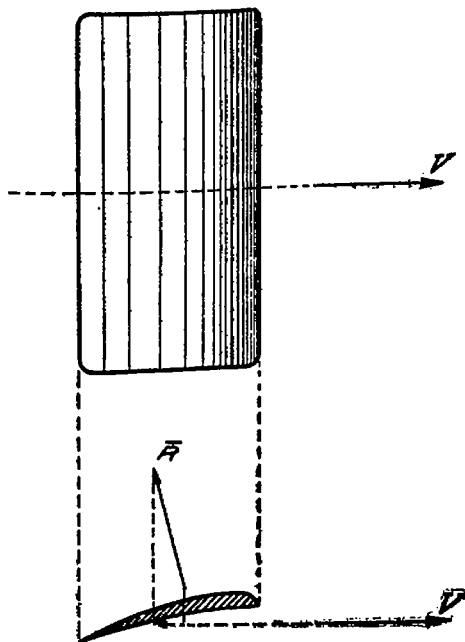


FIG. 3.

of the force of air resistance, the airplane had to be considered as suspended at that point when flying. This conception brought at once the false conclusion which is in full discordance with experience, that the lowering of the center of gravity would increase the stability, and this false conception was only the result of the consideration of the notion of point of application, which, exactly speaking, has nothing to do with the motion of rigid bodies. By the aid of the theorems of mechanics we can easily find at which point we can consider the airplane as suspended when in flight, so that our conclusions fully coincide with reality. This point is the center of mass, because the theorem of moments of momentum is applicable to the center of mass independently of its state of motion, so that the oscillation of a rigid body around its center of mass is the same as if the center of mass was immovable. From this right conception we see in full agreement with experience that the weight can have no influence on the stability of the airplane, which can be secured only by the forces of air resistance. I have stopped on the last question a little more than I ought to do, but the conception of center of pressure is generally so misunderstood in aviation that I thought that this explanation would not be unavailing.

Until now we have admitted that the fluid was immobile and the body moving in the fluid, but we could also consider the body as immobile and the fluid running by the body in a uniform stream. If in both cases the relative velocity of the stream toward the body is the same, the flow around the body can be the same in both cases, if the necessary precautions are taken for that purpose. But it can very easily happen that in these two cases the flow may be different because of differences in the boundary conditions. The principle of relativity of hydrodynamics consists in admitting that the fluid resistance depends only upon the relative velocity of the fluid to the body. It is clear that this principle can be admitted only when the flow around the body in both cases is the same; and under the latter conditions the principle of relativity is fully verified and is the conclusion of the general law of dynamics.

Very often the conception of point of application of a force is much misunderstood. Let us consider, for example, a nail fixed to a solid body, which is pulled by a rope fastened to that nail. In such a case we can certainly speak of the point of application of a force to our body, which point of application is the surface of contact of the nail with the body. When the nail is small and the body large, we can abstract ourselves in a first approximation from the size of the surface of nail contact and consider that surface as a point, and in that sense speak of the point of application of the force to the body. But as soon as we begin to speak about the equilibrium or the motion of our body, considering it as a solid body (by which we mean that we are neglecting the deformation of the body) under the action of the acting force, the conception of point of application loses every mechanical sense, as follows directly from our statement of the question, because, considering only the equilibrium or the motion of the body, we abstract ourselves from its other physical properties to which belongs also its elasticity. But the consideration of our body as invariable brings with it at once that the action of a force upon a body in sense of motion or equilibrium is independent of the position of the force on its line of action, so that from the standpoint of mechanics of rigid bodies only the magnitude of the forces and their lines of action have to be considered, the position of the forces on their lines of action being anyone, and we do not need to consider any point of application. Nothing astonishing must be found in that last fact. We must only remember the whole statement of the problem of motion of the rigid body. It must not be thought either that the consideration of such abstract concepts as the rigid body is something exclusive. On the contrary, one of the most important scientific methods consists in the sorting of the different sides of the questions studied by isolating by abstraction a physical property of a body from its other properties; and this general scheme of the evolution of our knowledge must never be forgotten, for doing so is the cause of great misunderstanding, as happens with the center of pressure in aviation.

In the beginning of the development of aviation, and by some writers until now, it was considered as evident that the center of pressure, being the point of application

CHAPTER II.

THE EMPIRICAL LAWS OF AIR RESISTANCE OF AEROFOILS.

Let us consider an aerofoil represented in cross section by figure 4, moving in air with a velocity constant in magnitude and direction. The air resistance \bar{R} of the aerofoil is fully specified by—

- I. Its magnitude;
- II. Its position and orientation toward the aerofoil.

To find the position and orientation of the air resistance \bar{R} as well as the orientation of the relative wind velocity \bar{V} toward the aerofoil let us take as reference line an arbitrary line LL invariably connected with the aerofoil cross section (see fig. 4). We will designate by α , and call it "angle of attack," the acute angle which the velocity \bar{V} makes with the line LL ; by β the angle which the air resistance \bar{R} makes with the normal to that same line, and by \bar{C} the point where the line of action of the air resistance \bar{R} cuts the line LL , which point will be called center of pressure. The orientation of the aerofoil relative to the velocity \bar{V} is fully specified by the angle α .

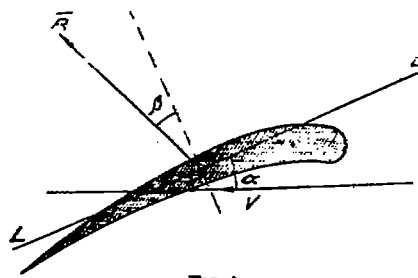


FIG. 4.

It has been shown by numerous experiments that the resultant air resistance encountered by an aerofoil, for certain intervals of the velocity variation, follows the following empirical law:

- I. In magnitude the air resistance \bar{R} of the aerofoil—
 - (1) Is proportional to the area A of the aerofoil;
 - (2) Is proportional to the square of the velocity V of the aerofoil relative to the air;
 - (3) Is a function of the orientation of the aerofoil toward the relative velocity V ;
 - (4) Is proportional to the air mass density δ .

II. In position and direction the air resistance of an aerofoil is independent of the magnitude of the velocity V and depends only upon the orientation of the aerofoil toward the relative velocity.

The foregoing empirical law of air resistance of aerofoils can be stated in the following formula:

$$R = k\delta A V^2 f(\alpha)$$

in which k is a coefficient of proportionality and $f(\alpha)$ a function of the angle of attack which is characteristic for the type of aerofoil considered. The last formula can also be written:

$$R = K A V^2 f(\alpha)$$

or

$$R = K_\alpha A V^2 = k_\alpha \delta A V^2$$

where

$$K = k\delta; K_\alpha = K f(\alpha) = k_\alpha \delta;$$

the coefficients K_α and k_α being certain functions of the angle of attack α only.

It is customary in aerodynamics to consider the resultant air resistance \bar{R} decomposed into two components, the drag R_x , along the relative velocity; \therefore

$$R_x = R \sin(\beta + \alpha) = K_\alpha A V^2 \sin(\beta + \alpha)$$

and the lift R_y , along the normal to the relative velocity.

$$R_y = R \cos(\beta + \alpha) = K_\alpha A V^2 \cos(\beta + \alpha)$$

when the air resistance R is in direction independent of the magnitude of the velocity V , we can write

$$\begin{aligned} R_x &= K_x A V^2 = k_x \delta A V^2 \\ R_y &= K_y A V^2 = k_y \delta A V^2 \end{aligned}$$

where

$$\begin{aligned} K_x &= k_x \delta = K_a \sin (\beta + \alpha) \\ K_y &= k_y \delta = K_a \cos (\beta + \alpha) \end{aligned}$$

the coefficients k_x and k_y being functions of the angle of attack α only.

Let us now examine the exact meaning of the foregoing empirical laws of air resistance and the restrictions to which these laws are submitted.

We shall begin by an exact statement of the definition of all the quantities which occur in the foregoing laws.

The angle of attack.—Let us designate by α_1 the angle of attack measured from one reference line $L_1 L_1$ and by α_2 , the angle of attack measured from a second reference line $L_2 L_2$, connected invariably both with the same aerofoil (see fig. 5); and let ϵ be the angle between these two lines. It is easy to see that we have

$$\alpha_2 = \alpha_1 + \epsilon$$

If the direction of V varies, we shall have

$$\alpha_2 + \Delta \alpha_2 = \alpha_1 + \Delta \alpha_1 + \epsilon$$

or

$$\Delta \alpha_2 = \Delta \alpha_1$$

because ϵ is a constant angle.

We therefore see that the variation of the angle of attack is the same for the same variation of the velocity orientation, independently of the reference line from which the angle of attack

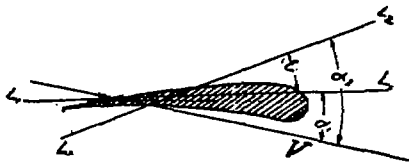


FIG. 5.



FIG. 6.

is measured. It is probably for the last reason that in the beginning of the development of aviation it was thought that the reference line used to fix the angle of attack can be chosen arbitrarily, and the *chord* of the aerofoil was generally adopted as such reference line. There would be nothing to say against such a convention if we had to do only with aerofoils with cross-sections of the same type, but all the difficulties begin when we wish to compare aerofoils with cross-sections of different profiles. It is in the conception of *chord* that the whole misunderstanding lies. In geometry the word *chord* is defined as a straight line joining two points of a curve, but what is the chord of an area like the section of an aerofoil? Nobody knows exactly, but, what is still worse, is that it is impossible to establish such a definition. When the cross-section profile of the aerofoil is formed by two curves which cut one another, we instinctively take as chord the common chord of the curves which limit the profile considered (see fig. 6a); but for profiles such as represented on fig. 6b two such chords can already be drawn. We are still more perplexed for the choice of the chord in the case such as shown in fig. 6c, in which any line drawn through that profile could with equal success be considered as chord. From these simple examples we see that the celebrated chord is nothing else than a reference line which is chosen arbitrarily. In such conditions when could we say that the profiles *a*, *b*, and *c* of the fig. 6 have the same angle of attack.¹ When we have to do with a flat plate the definition of the angle of attack presents no difficulty. It is evidently the angle between the relative

¹ This last question is of first importance for aviation practice. For example, how can we judge for two airplanes having their wings of different cross-section, that they are flying under the same angle of attack?

velocity and the plate itself (see fig. 7). But what is the aerodynamical characteristic of the direction of the flat plate? It is nothing else than the direction for which the lift of the plate is zero. When the wind blows along the plate, the whole air resistance is reduced to drag and we have no lift. It is consequently from the direction of zero lift that we measure the angle of attack of a flat plate. Thus the direction of zero lift forms our reference line in that case.

Many years ago Paul Painlevé indicated that, if we wish to obtain a rational basis for the establishment of the definitions of all the conceptions which we use in connection with the air resistance law of aerofoils, we must simply draw a parallel between the aerofoil and the flat plate considered as a conceptional standard.

Adopting this standpoint, we shall adopt as reference line of each aerofoil the direction for which its lift is zero and we shall call that line the *zero lift line* or, more simple, the *zero line*. The plane normal to the symmetry plane of the aerofoil and containing the zero line will be called the *zero plane*. The zero plane and the zero line are experimentally fully determined for each aerofoil.

Let us consider an aerofoil (see fig. 8) on which the wind blows successively in the directions V_1, V_2, V_3, V_4 and let R_1, R_2, R_3, R_4 be the air resistance corresponding to those directions. We reach the zero line when the resultant air resistance is in the wind direction, as is the case for R_4 . The zero line has to be determined experimentally not only in direction but also in *exact position* relatively to the aerofoil.

It is easy to see that for each type of aerofoil we generally have four zero lines as shown on figure 9. We shall adopt as *standard zero line* the one which corresponds to zero lift when the wind is blowing on the *entering edge*. It is the zero line which corresponds to V_1 and R_1 in figure 9. The angle of attack measured from the standard zero line will be designated by i and called *absolute angle of attack* or *absolute incidence* (see fig. 10); distinguishing this angle from the relative angle of attack α measured from any other reference line. The standard reference line and absolute angle of attack as above defined are important aerodynamic characteristics of the aerofoil.¹

The partisans of chord have reproached the definition of the standard reference line with the fact that it is difficult experimentally to measure the incidence from that line. But it is quite another question when we have to determine in experimentation the orientation of aerofoils in the wind current. In that case we certainly must choose as reference line that line from which the measurements are most easily made and such a line could be called the *experimental reference line*. The question of experimental reference line is a question of the technic of experimentation. In one experimental method, one line is more convenient; in another method, another line is more convenient. But when stating the results of our experimentation, we must always give them in absolute angle of attack, because only in this case will comparison be possible.

Finally I must also mention the following fact: It can happen that for a certain aerofoil cross-section the lift may be zero for any direction of the relative wind within a certain angle, as shown in figure 11. In that case one of the extreme zero lines, V_1R_1 or V_4R_4 of the above figure, ought to be taken as reference line. In such a case, the lift curve plotted, for example, as function of the incidence would have the shape shown in figure 12.

The aerofoil area.—The area of an aerofoil also needs a special definition. According to our standpoint of a parallel drawn between aerofoil and flat plate, we shall adopt as "aerofoil area" the area of the projection of the aerofoil on its zero plane. (See fig. 10.) Only with such a definition will be avoided all the difficulties and indeterminations, as will be easy to see from the detailed discussion which has been made for the angle of attack.

The center of pressure.—To avoid difficulties, we must also adopt as center of pressure the point of intersection of the zero line with the resultant force of air resistance \bar{R} .

¹ For example, an absolute incidence of five degrees means that a five degree decrease of the angle of attack bring us to zero lift. For the aerofoils actually used in aviation practice the standard zero line is generally disposed above the aerofoil, which means that when the absolute incidence is equal to zero the air resistance gives rise to a moment relative to the entering edge.

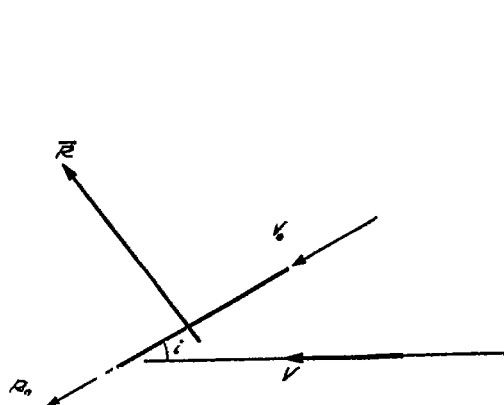


FIG. 7.

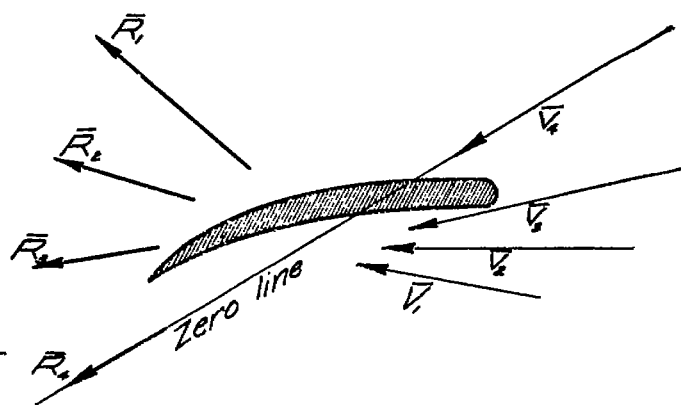


FIG. 8.

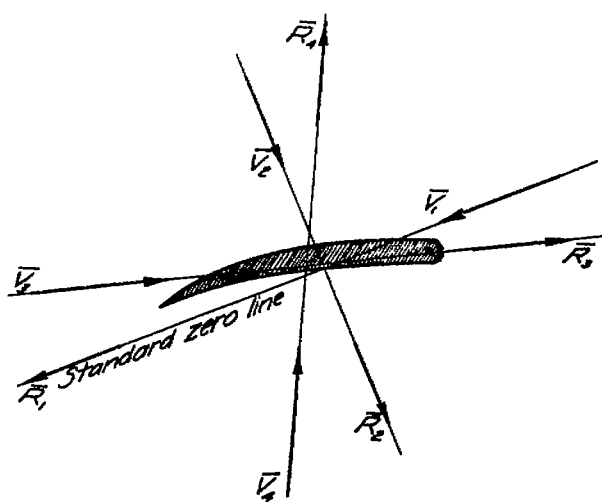


FIG. 9.

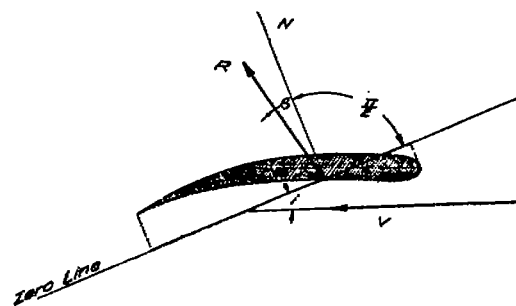


FIG. 10.

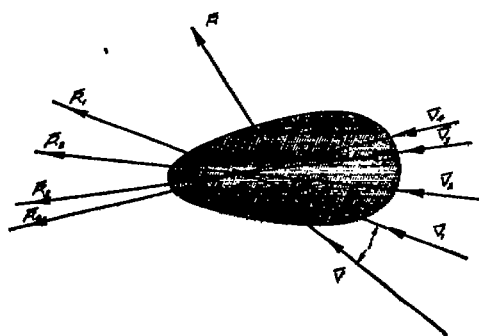


FIG. 11.

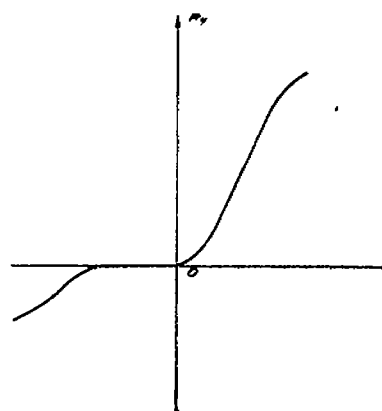


FIG. 12.

I shall show in a few words what, for example, happens if we take for the center of pressure the intersection of the so-called chord and the air resistance \bar{R} . In figure 13 is represented an aerofoil and the system of the resultant forces of air resistance $R_0, R_1, R_2, R_3, R_4, R_5$, for different angles of attack increasing in an arithmetical progression. If we follow the displacement of the center of pressure O' referred to the chord, we easily see that there is one position of the air resistance when it is parallel to the chord, and the center of pressure goes to infinity. So that the curve of the center of pressure taken on the chord has always for small angles of attack an asymptote. The general shape of the curve of center of pressure in that case is shown in figure 14a, where is plotted the distance of the center of pressure O' from the entering edge A' as function of the relative angle of attack α' . If we take the center of pressure on the zero line, then the curve of center of pressure will not have any point at infinity and the curve of center of pressure will have the shape represented in figure 14b¹, where is plotted the distance AC (see fig. 13) as function of the absolute incidence i . The passage of the center of pressure to infinity when taken on the chord is only a consequence of a bad definition, because it must be remembered that in the definition of the center of pressure we must be guided only by convenience.

To illustrate fully the meaning of the conception of the center of pressure, I shall draw a parallel between the notions of *center of pressure*, *center of mass*, and *metacenter*.

Let us first consider two parallel forces F_1 and F_2 of constant magnitude applied at two points 1 and 2. (See fig. 15.) As well known, the resultant R_{12} of these two forces will be parallel to them and will divide the distance 1, 2 in inverse ratio to the forces F_1 and F_2 . If we consider now the two forces F_1 and F_2 turning around their points of application but maintaining their magnitude and remaining parallel, the resultant force R_{12} will also turn around a definite point. If we consider now a system of three parallel forces F_1, F_2, F_3 constant in magnitude and turning around their points of application, it will be easily seen that the resultant force of the three forces will also turn around a definite point, because R_{12} is the resultant of R_{12} and F_3 , and so on, independently of the number of forces. The point through which the resultant force of a system of constant parallel forces turning around their points of application always passes, is called the *center of the parallel forces*. The center of mass is a particular case of center of parallel forces when the forces considered are the weights of the different elements of a body.

Let us now consider generally any system of forces applied at any points. If we consider the continuous variation of these forces, their resultant force will also vary continuously in magnitude, position, and direction, and will describe in space a certain surface which is called the *metacentric surface*. When all the forces considered lie in the same plane, the resultant force also lies in the same plane and the metacentric surface is reduced to a *metacentric curve*, which is the envelope of the successive positions of the resultant force. The point at which the resultant force touches the metacentric curve is called the *metacenter*. (See fig. 16.) When the forces considered are parallel and constant in magnitude, the metacentric curve reduces to a point. We therefore see that we can consider the center of mass as a particular case of metacentric curve reduced to a point.

If we consider the system of forces of air resistance of an aerofoil, these forces admit a metacentric curve and it will be easy to see that this metacentric curve has always a cusp point admitting the zero line as tangent at that point. In figure 17 is represented the general shape of the metacentric curve of an aerofoil. For comparison, in figure 18 is represented the metacentric curve of a flat plate.²

The important fact is that the center of pressure is neither a center of parallel forces nor a metacenter, but simply a point arbitrarily chosen to fix the position of the resultant force

¹ The scale used on fig. 14a to plot the distance $A'O'$ is smaller than the scale used on fig. 14b to plot AC .

² In the theory of the airplane the metacentric curves do not have the same importance as in ship theory. In the last theory the metacentric curves allow a direct evaluation of the restoring moments, on account of that fact that, to a first approximation, the lifting force of a ship is constant when the ship undergoes oscillations. It is not the case for the airplane, where the lifting forces are variable in magnitude when the airplane is oscillating, so that the metacentric curve alone does not determine the restoring moment. That is why for aerofoils the metacentric curve must be considered only as giving the general picture of variation of the resultant force of air resistance in position and direction.

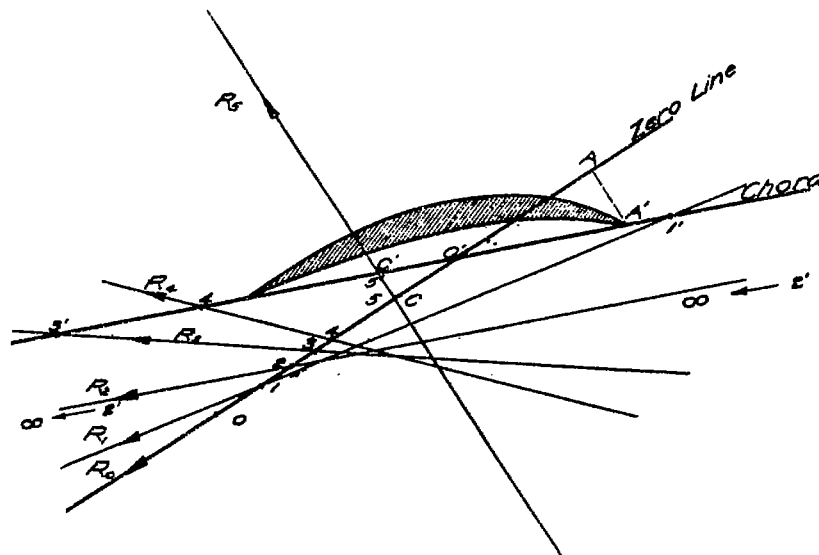


FIG. 14.

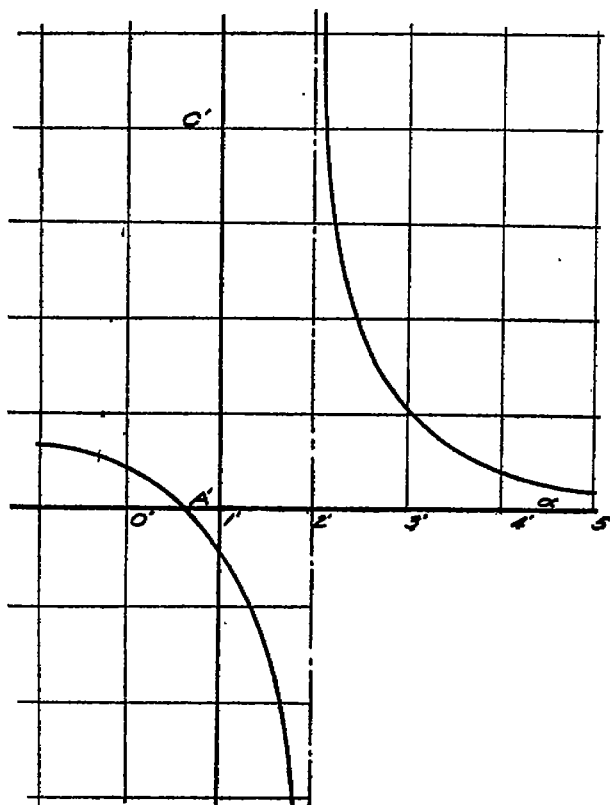


FIG. 14a.

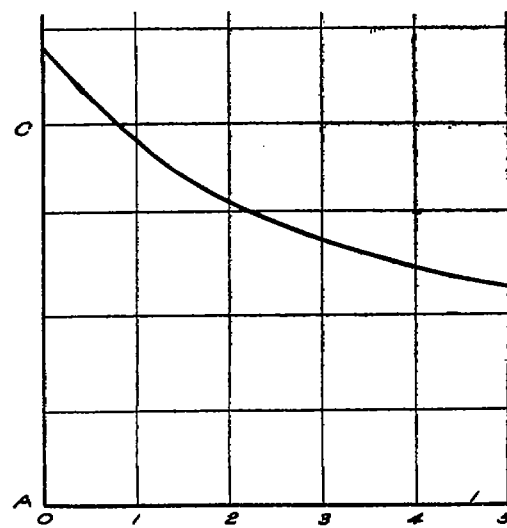


FIG. 14b.

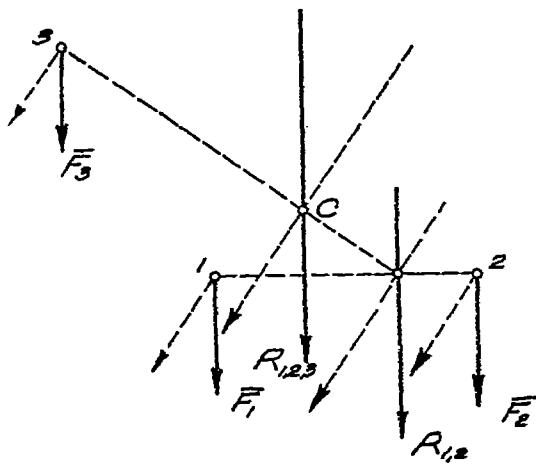


FIG. 15.

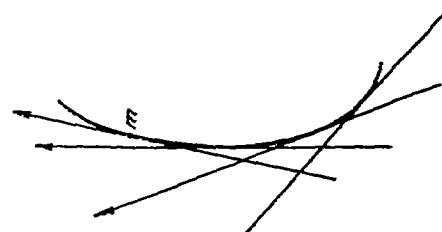


FIG. 16.

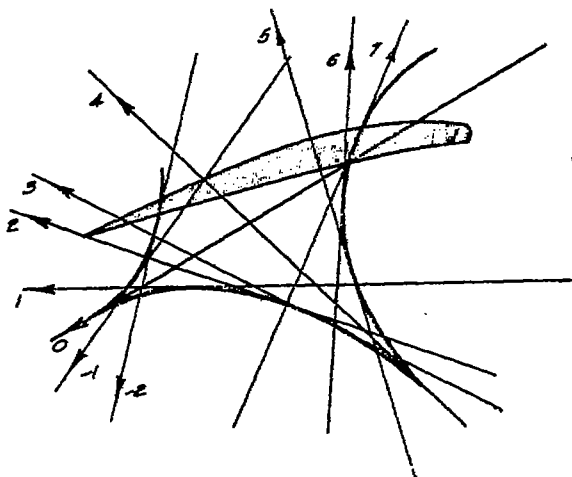


FIG. 17.

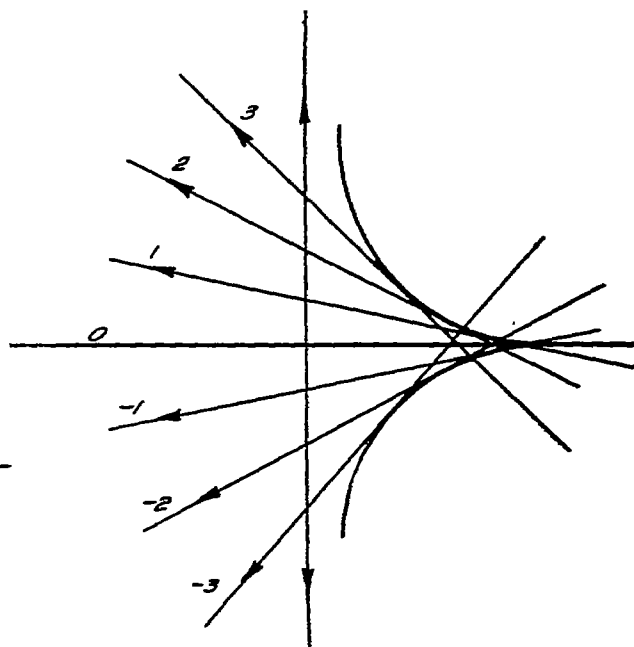


FIG. 18.

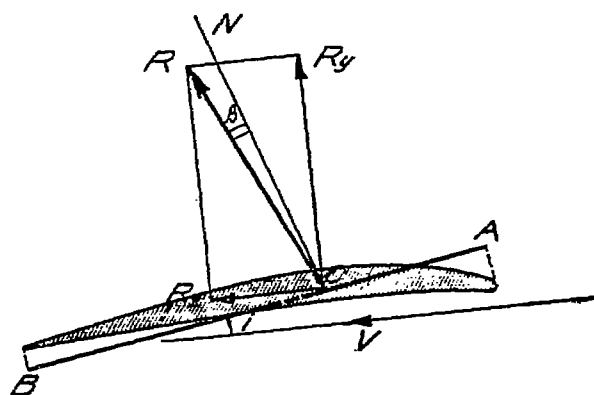


FIG. 19.

of air resistance. That is why we must choose it in the way most convenient for our purpose. For this last reason we shall take the center of pressure on the zero line. In that case we shall have no point of the center of pressure curve in infinity, and the cusp point of the meta-centric curve will be the limit position of the center of pressure on our zero line.

The velocity.—The velocity V which enters in the formulas of air resistance of aerofoils disposed in a uniform, fluid current has to be taken in front of the aerofoil and at such a distance from it that the disturbances in the medium caused by the presence of the aerofoil do not reach it. This distance generally lies in front of the aerofoil between one and two times its breadth.

The fictitious equivalent plane.—As a summary of all the foregoing discussion we are brought to the following conception:

Let us consider the zero plane of an aerofoil and project on this plane the aerofoil and take this area as the *fictitious equivalent plane*, or, shorter, as *equivalent plane*, of our aerofoil; that is to say, attribute all the properties of our aerofoil to that plane and refer all the quantities which we use to describe the law of air resistance of aerofoils to that fictitious equivalent plane. We shall thus take as area of the aerofoil the area of the equivalent plane. (See fig. 19.) We shall measure the angle of attack from that equivalent plane and this will be our absolute incidence i . We shall take the center of pressure on that equivalent plane and fix the direction of the force of air resistance R by the angle β of its inclination to the normal to that equivalent plane. Under such conditions, all the formulas of pages 2 and 3 have to be referred to the equivalent plane; and in that case we shall write:

I. The magnitude of the air resistance \bar{R} of aerofoils (see fig. 19)

$$R = k\delta A V^2 f(i) = K A V^2 f(i)$$

$$R = K_1 A V^2 = k_1 \delta A V^2$$

where

$$K = k\delta; K_1 = K f(i) = k_1 \delta$$

The drag

$$R_x = R \sin (\beta + i) = K_x A V^2 = k_x \delta A V^2$$

The lift

$$R_y = R \cos (\beta + i) = K_y A V^2 = k_y \delta A V^2$$

where

$$K_x = k_x \delta = K_1 \sin (\beta + i)$$

$$K_y = k_y \delta = K_1 \cos (\beta + i)$$

the coefficients K_x , K_y and k_x , k_y being functions only of the absolute angle of attack i .

Some general data on aerofoils.—To the foregoing formulas I will add the following remarks:

For the orientation under which the aerofoil is practically used, the lift of the aerofoil is generally equal to zero only when the wind is blowing on the back of the aerofoil, and the equivalent plane is disposed somewhat above the aerofoil. The position and orientation of the equivalent plane can, in general, also depend from the value of the speed V , so that to different speed intervals can correspond, for the same aerofoil, different equivalent planes.

Starting from zero absolute incidence, the air resistance \bar{R} rises very quickly out of the zero plane, so that for angles of attack around 5° , the air resistance makes small angles with the normal to the zero plane.

For the aerofoils actually used in aviation for small angles of attack, the ratio of drag to lift can reach $1/20$.

For actual aerofoils, considering the incidence increasing from zero, the center of pressure first approaches the leading edge (see fig. 17)—that is, travels in a sense inverse to that for the case of a flat plate (see fig. 18)—and only afterwards, for greater values of the angles of attack (generally larger than 10°) the center of pressure begins to travel away from the leading edge.

The coefficients K_x , K_y , and k_x , k_y , for equal values of the angle of attack, have the same values only for aerofoils having similar cross-section and similar perimeters; and still in that

case their values also depend upon the magnitude of the aerofoil area A and the magnitude of the velocity V . These coefficients do not vary much when the magnitude of the aerofoil area changes, and by the variation of the velocity the coefficients K_x and k_x are principally affected—they diminish when the velocity increases—the coefficients K_y and k_y do not seem to depend much upon velocity for a value of the last above a certain value.

For small angles of attack—up to around 10° —for most actual aerofoils, the coefficients k_y follows a linear law and the coefficient k_x a parabolic law. So that for such angles we can write

$$\begin{aligned} k_y &= ki \\ k_x &= k(ai^2 + bi + c); \end{aligned}$$

so that in such a case the lift and drag of the aerofoil have for expressions

$$\begin{aligned} R_y &= k\delta A V^2 i = KA V^2 i \\ R_x &= k\delta A V^2 (ai^2 + bi + c) = KA V^2 (ai^2 + bi + c) \end{aligned}$$

The value of the coefficient K depends upon the *aspect ratio* L/b , that is, the ratio of its span L to its breadth b . For values of this ratio equal to about 5 or more, the coefficient K for most actual aerofoils, for usual atmospheric conditions, has a value near to $1/200$, the units used being the meter, the kilogram, and the second. For smaller values of the aspect ratio, the value of K diminishes.

Different characteristic curves used to plot the results of measurements of the air resistance of aerofoils.—To plot the results of measurements of air resistance of aerofoils different systems of curves are used. From any system of characteristic curves giving a full specification of the laws of air resistance of aerofoils, we can deduce any other one.

*First method.*¹—The most direct way of representing the air resistance of an aerofoil is to plot the curves of the coefficients K_i or k_i as function of the angle of attack i , and the curve of the angle β as function of the angle of attack i . The K_i or k_i curve gives a direct evaluation of the magnitude of the force of air resistance; and the β curve gives the laws of variation of the inclination of the air resistance to the normal to the zero line. The general shape of the K_i and β curves are represented in figure 20.

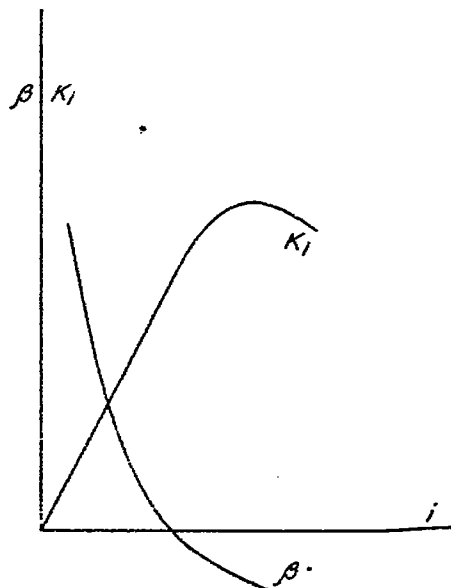


FIG. 20.

The general shape of the K_i and β curves are represented in figure 20.

Second method.—Another method very widespread in the practice of modern aerodynamical laboratories is to plot the lift curve K_y and the drag curve K_x as functions of the angle of attack i . To these curves the drag-lift K_x/K_y curve is generally added. It is much more convenient to use the drag-lift curve than the lift-drag curve, as it is made sometimes, because many fundamental properties of the airplane are directly connected with the drag-lift curve.² The general shapes of the drag curve, the lift curve, and the drag-lift curve are represented in figure 21.

Third method.—Probably one of the oldest methods used to represent the laws of air resistance of aerofoils consists in plotting the lift coefficient as function of the drag coefficient. This method was used by Lilienthal. When using this method the angle of attack is marked on the curve. (See fig. 22.) This method presents the advantage that we can also read on the curve $K_y = F(K_x)$ the variation of the drag-lift ratio. It is easy to see that the tangent

¹ This method of plotting is used by the author for propeller calculations.

² See, for example, G. de Bothezat, "Étude de la Stabilité de l'Aéroplane," pp. 58-68, Paris, 1911.

of the angle γ which a line joining the origin with a point of the curve $K_y = F(K_x)$ makes with the K_y axis is equal to

$$\operatorname{tg} \gamma = \frac{K_x}{K_y};$$

so that we can directly plot a scale for K_x/K_y on a parallel to the K_x axis. Each straight line joining the origin with a point of the $K_y = F(K_x)$ curve cuts off on that scale the value of K_x/K_y . The tangent drawn from the origin to the curve $K_y = F(K_x)$ gives the minimum of the value of K_x/K_y .

In his last research on aerofoils, Eiffel uses this method and for convenience plots the K_x at the scale ten times bigger than the K_y .

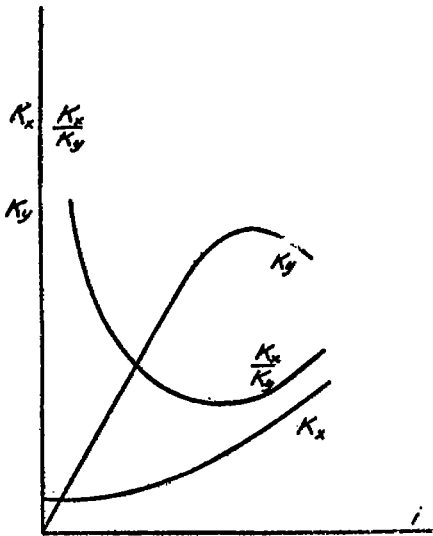


FIG. 21.

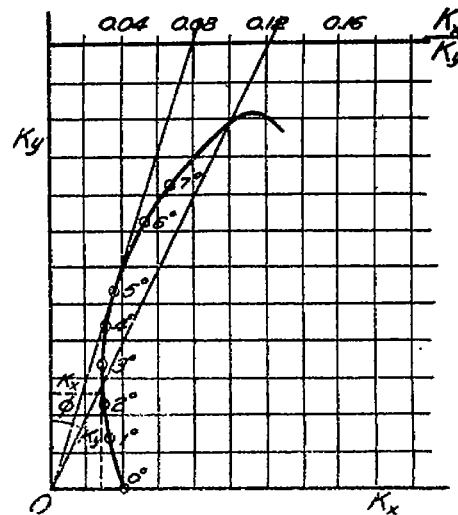


FIG. 22.

To specify fully an aerofoil by each system of the foregoing curves, there must be added the curve of the center of pressure and the zero line in exact position and direction. It is also good to draw the metacentric curve which gives a full picture of the positional and directional variation of the forces of air resistance.

I must also add that it is necessary that the data on aerofoils be at least determined for an interval of -90° to 90° of absolute incidence. This is on account of the fact that we must not limit ourselves to the actual necessities, but must also give data which future research and discovery may need.¹

¹ As example, I can indicate the following fact: Only because the aerofoil data were not enough extended, we can not actually calculate the thrust of a propulsive screw at a fixed point whenever the necessary methods are already at our disposal. A propeller at a fixed point works at very large angles of attack, 20° , 30° , 40° , and in some cases still greater.

CHAPTER III.

THE FLOW AROUND THE AEROFOIL.

In the beginning of the development of aviation, the main knowledge arose at first only to the general quantitative relations about air resistance of aerofoils, and it is only very slowly that light has been thrown on the flow phenomena.

For an aerofoil moving with a uniform and rectilinear velocity in air the following flow phenomena have been progressively discovered experimentally.

I. On the upper surface of the aerofoil we have a decrease of pressure and on the lower surface of the aerofoil we have an increase of pressure. The depression created on the upper surface is, for small angles of attack, always larger than the increase of pressure on the lower surface, so that the *lift of the aerofoil is due more to a suction exerted on the upper side than to the pressure exerted on the lower side.*

II. The stream in the wake behind the aerofoil *appears* to be deflected downward.

III. From the tips of the aerofoil vortices run off which we will call the *tip vortices* (see Fig. 23). The rotation of the fluid in these tip vortices has the sense from the outside space into the inside space between the vortices, if we look from above.¹

IV. In the space between the tip vortices two kinds of flow can take place. For very small angles the flow is continuous; that is to say, we have no sensible turbulent motion.

But when the angle of attack increases beyond a certain value of the last, there appear on both edges of the aerofoil vortices, parallel to these edges, which we will call the *edge vortices*. The greater the velocity of the flow running on the aerofoil, the smaller is the angle of attack for which edge vortices appear. These edge vortices are not stationary with reference to the aerofoil. They grow up on the edges of the aerofoil and, when they have reached a certain intensity, they run off in the general direction of the stream behind the aerofoil, so that these edge vortices have a certain velocity with reference to the aerofoil.

The edge vortices which grow on the upper and lower edge rotate from the space outside the two vortices into the space inside, when one looks from above, so that behind the aerofoil there appears a system of vortices in quincunx² rotating in inverse senses (see fig. 24). The ends of these vortices go over into the tip vortices. As the edge vortices are rotating in inverse sense the mean value of the intensity of the tip vortices is not modified by the edge vortices. We now see that the general picture of the flow behind an aerofoil looks like a vortex ladder running off the aerofoil.³

So far as I know, exact measurements of the depression on the upper side and the pressure on the lower side of an aerofoil were first made by G. Eiffel. The apparent stream deflection behind the aerofoil seems to have been observed by many investigators. The necessity of the existence of tip vortices seems to have been first indicated by Lanchester. The vortices in

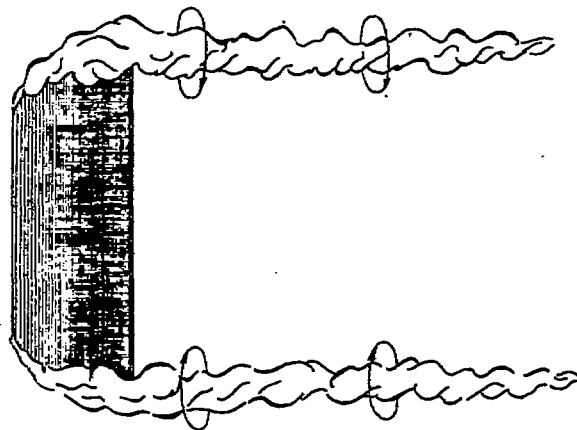


FIG. 23.

¹ Observations of tip vortices have been made by J. R. Pannell and N. R. Campbell. "The Flow of Air Around a Wing Tip," Report No. 197, March, 1916, Advisory Committee for Aeronautics.

² I call *quincunx vortex system*, a system of two parallel rows of equidistant rectilinear and parallel vortices rotating in inverse senses in each row, and in such an arrangement that the vortices of one row are disposed towards the middle of the distance between the vortices of the other row.

³ Good pictures of edge vortices running off an aerofoil can be found in the "Technical Report of the Advisory Committee for Aeronautics," for the years 1912-1913. "Photographic Investigation of the Flow Around a Model Aerofoil," by E. F. Belf, p. 133.

quincunx were first noted by Karman for the particular case of the orthogonal motion of a flat plate and the motion of a cylinder.

Before attacking the detailed discussion of the above-mentioned flow phenomena, I shall first make some general remarks on the flow phenomenon around a solid body moving in a fluid with a uniform velocity.

General scheme of the phenomenon of fluid resistance.—I shall here develop a general scheme of the fluid resistance phenomenon, which must be considered as a conceptional limit, but which takes account of all the fundamental circumstances of the fluid resistance phenomenon in their most simplified form, and thus allows a better view of the relations which hold.

We shall first consider the case of an infinite cylindrical body having a plane of symmetry and moving in an infinite fluid with a constant velocity V_0 parallel to that plane of symmetry, the sense of V_0 being taken as positive sense. (See fig. 25.)

Let us imagine an observer moving with the body. For such an observer there will appear

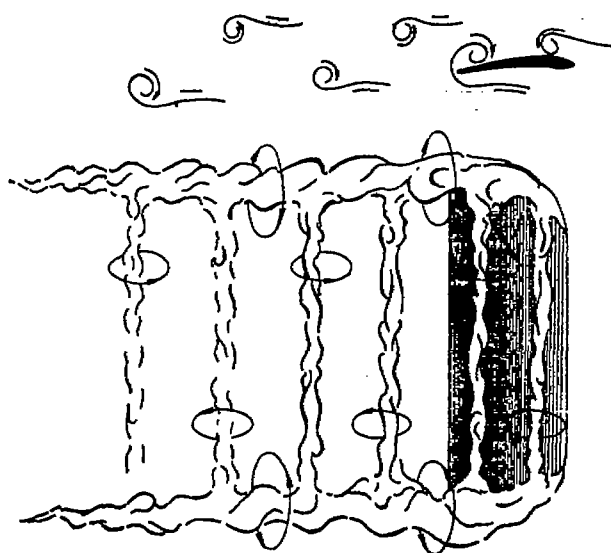


FIG. 24.

a relative stream running on the body. When the fluid is considered as perfect this relative stream can be assumed as being a potential stream—that is, a stream admitting a velocity potential for the velocity distribution in it. But for a real fluid, in the case of our problem, according to the indications of the experiment, there must necessarily be losses inside the fluid and thus a certain distribution of vortices in it. This last fact is a direct consequence of the general equations of motion of a viscous fluid, according to which there are no losses inside the fluid where there are no vortices.¹ For the general analysis of the fluid resistance phenomenon we will place ourselves in ideal limiting conditions and replace the effective relative stream running on the body by a *conventional relative stream*, but

so defined that in relation to the fluid resistance the conventional relative stream will be fully equivalent to the effective relative stream.

We will first assume that in each cross section normal to the plane of symmetry of our body the velocity of the conventional relative stream is constant. In such conditions, to take account of the change in the distribution of the velocities in the general stream which are produced by the presence of the body, we must consider our conventional relative stream as limited by surfaces of discontinuity outside which the general stream velocity is unmodified, but inside which the velocity, being constant in each cross section, is different from the outside velocity. (See fig. 25.) These surfaces of discontinuity must thus necessarily be constituted by vortex sheets. On the other hand, as we must also conceive the fluid as adhering to the surface of the body—a fact to which seem to lead Zahm's² experiments on skin friction, which have shown its independence of the state of the body's surface—we must consider the surface of the body as covered by a vortex sheet in which gliding of the fluid takes place, the relative velocity at the surface of the body being equal to zero. We thus see that in our conception of the flow phenomenon around the body the vortices instead of being spread in a certain way inside the

¹ According to Lamb, "Treatise on the Mathematical Theory of the Motion of Fluids," the dissipation of energy inside a fluid mass is given by the general expression

$$4\mu \iiint (\omega_x^2 + \omega_y^2 + \omega_z^2) dx dy dz$$

which is equal to zero for $\omega_x = \omega_y = \omega_z = 0$

² "Atmospheric Friction on Even Surfaces," Philosophical Magazine, July, 1904.

fluid have to be conceived as concentrated on the surface of the body and on the boundary of the conventional relative stream. (See fig. 25.)

Let us consider now two cross sections, I and II, of the relative stream running on the body. Both cross sections are considered immobile relative to the body and are taken normal to the stream. Section I, of conventional height h_0 , is taken before the body at a distance not reached by the disturbance created by the body in the fluid. The relative velocity and the pressure in that section, uniform in the whole section, are designated by p_0 and $-V_0$, this last velocity being equal in magnitude to the velocity of the body, but having an inverse sense. The Section II is taken behind the body; p and $-V$ are the uniform pressure and relative velocity in that section. The absolute velocity w of the stream behind the body is equal to

$$(1) \quad w = -V - (-V_0) = V_0 - V.$$

The velocity w is nothing but the mean velocity of the wake behind the body.

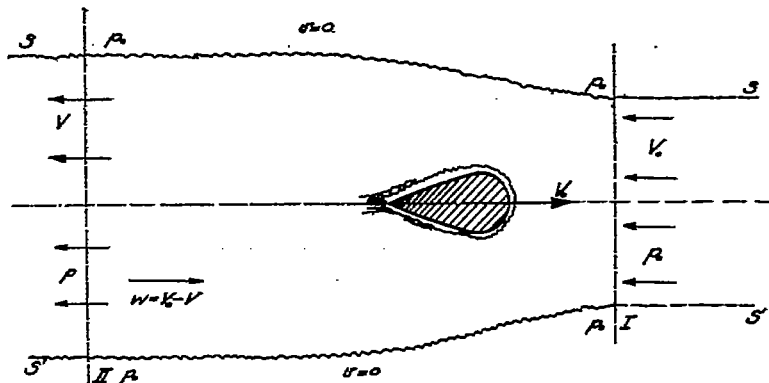


FIG. 25.

Let us designate by C the value of the Bernoulli constant corresponding to Section I. According to Bernoulli's theorem we must have

$$(2) \quad p_0 + \frac{\delta V_0^2}{2} = C.$$

Let us follow from Section I to section II a stream line in the relative motion of the fluid toward the body. Starting from the values p_0 and V_0 in Section I, pressure and velocity will vary along the stream line. When we pass by the body, the velocity will be increased and the pressure decreased. Behind the body the velocity will drop and the pressure increase, and when we reach Section II we shall find there a pressure p and a velocity $-V$ connected by the relation:

$$(3) \quad p + \frac{\delta V^2}{2} = C - \Delta C,$$

where ΔC is the drop in the Bernoulli constant, which occurs when we go from Section I to Section II. In reality this drop is due to the losses taking place at the surface of the body by skin friction and inside the fluid by viscosity, which losses, in our limited conception, are assumed to be concentrated on the boundaries of our conventional stream. Subtracting equation (4) from equation (3), we obtain the relation connecting p_0 and V_0 in Section I with p and V in Section II.

$$(4) \quad p_0 - p + \frac{\delta}{2} (V_0^2 - V^2) = \Delta C.$$

Let us designate by Π the whole amount of work done by the forces of viscosity inside the fluid between the Sections I and II. We can always consider this interior work referred to the velocity V_0 in Section I and consequently write

$$(5) \quad \Pi = F V_0,$$

where F is a fictitious force, which we call the dissipative force and which corresponds to the interior work done between the Sections I and II and referred to the velocity V_0 . By its direct meaning, ΔC is the interior work by unit of volume of the relative flow crossing the two sections I and II in a second, which volume, counted per unit of length of the body, as a consequence of continuity, is equal to either of the two expressions

$$(6) \quad h_0 V_0 = h V,$$

when one neglects the very small density variations. We thus have

$$(7) \quad \Delta C = \frac{\Pi}{h_0 V_0} = \frac{F V_0}{h_0 V_0} = \frac{F}{h_0}$$

or

$$(8) \quad F = h_0 \Delta C.$$

The general picture of the flow around the body begins now to appear more clearly. Outside the boundaries SS and $S'S'$ of our conventional relative stream (see fig. 25), as we consider the fluid unaffected by the motion of the body, we thus have the uniform pressure p_0 ; and in the absolute motion the fluid is immobile, so that in Section II from outside to inside we have a difference of pressures $p_0 - p$ and a difference of velocities $w = V_0 - V$, maintained by the conventional boundaries SS and $S'S'$, which are vortex sheets.

Let us now apply the theorem of momentum to the fluid mass contained between the body considered, the vortex sheets SS and $S'S'$, and the Sections I and II and included between two planes normal to the body at a unit distance from one another. Let us designate by $R_x = k_x \delta b V_0^2$ the drag of the body counted per unit of length, b being a linear dimension of the body. This drag constitutes, in our case, the resultant of all the forces acting on the surface of the body. It will be easy to see that we have

$$(9) \quad \delta h_0 V_0^2 - \delta h V^2 + h(p_0 - p) = k_x \delta b V_0^2 (*)$$

This last relation in connection with the relations

$$(10) \quad (p_0 - p) + \frac{\delta}{2}(V_0^2 - V^2) = \Delta C = \frac{F}{h_0}$$

and

$$(11) \quad V_0 h_0 = V h$$

gives thus three equations connecting the pressure p , the velocity V and height h in the section II with the corresponding quantities p_0 , V_0 , and h_0 in the section I. From all these quantities the only ones to be considered as known in our problem are p_0 and V_0 .

It is easy to show that when the section II is considered taken at such a distance from the body that either $p = p_0$ or $V = V_0$, we will have very approximately

$$(12) \quad F \approx k_x \delta b V_0^2$$

That is, the dissipative force equals the drag.

For when $V = V_0$, which brings with it $h = h_0$, we have

$$(13) \quad F = h_0(p_0 - p) = h(p_0 - p) = k_x \delta b V_0^2$$

and when $p = p_0$, then

$$(14) \quad \begin{aligned} F &= \frac{\delta}{2} h_0 (V_0^2 - V^2) \\ &= \frac{\delta}{2} h_0 V_0^2 - \frac{\delta}{2} h_0 V^2 + \frac{\delta}{2} h_0 V_0^2 - \frac{\delta}{2} h_0 V_0^2 \\ &= \frac{\delta}{2} h_0 V_0^2 - \frac{\delta}{2} h V^2 \left(\frac{h_0}{h} + \frac{h_0 V_0^2}{h V^2} \right) \\ &= \frac{\delta}{2} h_0 V_0^2 - \frac{\delta}{2} h V^2 \left(\frac{h_0}{h} + \frac{V_0^2}{V^2} \right) \end{aligned}$$

* To find the resultant of the outside pressure on the boundary of the portion of the relative stream considered, it is sufficient to conceive the pressure p_0 added and subtracted in the section II. We then will have a uniform pressure p_0 on all the boundary surface, whose resultant is zero, and the pressure $h(p_0 - p)$ in the section II, which quantity constitutes the resultant pressure.

But, as will be seen in the following, when k_x is a small quantity, the difference between h_0 , h , and V_0 , V in case of $p = p_0$ is negligible, so that

$$(15) \quad F \approx \delta h_0 V_0^2 - \delta h V^2 = k_x \delta b V_0^2$$

In the general case

$$(16) \quad F < k_x \delta b V_0^2$$

The relations (15) and (16) lead us to make the assumption that the dissipative force can be considered as having the form

$$(17) \quad F = f \delta b V_0^2$$

where f is a characteristic coefficient depending upon the form of the body and the properties of the fluid and the position of section II. In the case of $p = p_0$ or $V = V_0$ we have

$$f = k_x$$

But in the general case we will have

$$f < k_x$$

The inferior limit for f is determined by the skin friction at the surface of the body.¹

All the foregoing constitutes, so to speak, a limited scheme of the fluid resistance phenomenon, but one which gives a complete picture of the relations occurring. Let us examine the connection between our scheme and reality.

In the relative flow around a body, the observed velocities in a section such as section I are uniform when the section is taken at a sufficient distance from the body, but in a section such as II the uniformity of velocity is generally not observed. This last fact does not constitute an essential difference, because we can always conceive the velocity V as a certain mean value of the real velocities.

Much more essential is the question of the practical possibility of the existence of the vortex sheets at the surface of the body and on the boundary of the stream. It has been pointed out by many investigators that vortex sheets in viscous fluids must be considered as unstable.² Experiments performed on the observation of the flow around bodies, although not very numerous, have already given valuable indications.³ For relative flow velocities having a sufficient value, the vortex sheet covering the surface of the body always passes over into a system of vortices in quincunx. This last fact was first fully understood by Karman, who also indicated the reason why we get the quincunx vortex system. Karman's investigations of the quincunx vortex system have shown that this system is stable. The edge vortices above mentioned are nothing else than the vortices in quincunx into which the vortex sheet covering the surface of the body passes. For low velocities we also have in all probability a tendency toward the formation of the vortices in quincunx, but the energy in the wake being small, the energy of the beginning vortices is dissipated before their complete formation. The motion which is established must be a kind of turbulence which distributes inside the fluid the vortex sheets covering the surface of the body. The mechanism of this distribution is in all probability the following: We either have a direct, irregular, and periodical transformation of the surface vortex sheet in quincunx vortices, dissipated before full formation, reformation of the surface vortex sheet, and so on; or we have a periodical irregular formation of the surface of discontinuity established in the Kirchhoff-Lord Raleigh theory. These surfaces of discontinuity, which must necessarily be vortex sheets, can appear as inside boundaries of the relative motion, only as momentary phenomenon. At such a moment the flow appears as represented in figure 26. But these surfaces, being unstable, quickly disappear and the vortex intensity concentrated in them is dissipated before the formation of a definite

¹ It must be remarked that the calculation of the resultant skin friction at the surface of a body often presents some ambiguity, the exact distribution of the velocity at the surface of the body having to be known.

² See H. Poincaré, "Théories des Tourbillons," p. 174, § 142; A. B. Basset, "A Treatise on Hydrodynamics," Volume II, p. 309; and O. Schaefer, "Einführung in die Theoretische Physik," Band I, p. 806, § 189.

³ For a review of these observations see W. L. Cowley and H. Levy, "Aeronautics in Theory and Experiment," Chap. II.

kind of vortex motion. Such irregular, unstable, periodical process, affected by the smallest perturbation, spreads the surface vortices in the fluid.

In whole probability all the observations made upon the vortex sheets surrounding the body apply also to the vortex sheets SS and $S'S'$, constituting the outside boundaries of our conventional relative stream. For small relative flow velocities, those vortex sheets are in a certain way distributed in the fluid; but for greater velocities it is possible that they go over into the stable system of vortices in quincunx. We thus see that behind a body moving in a fluid we shall have, in general, a periodical fluid motion. I shall call *primary or fundamental wave* the fluid motion generated by the formation of quincunx vortices from the vortex sheets limiting the

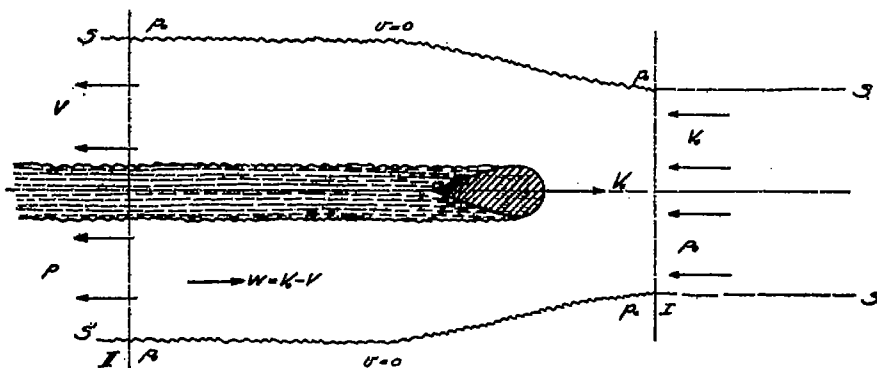


FIG. 26.

relative stream boundaries from the outside, and *secondary wave* the fluid motion generated by the formation of quincunx vortices from the vortex sheets limiting the inside relative stream boundaries. Both fundamental and secondary waves will be considered in more detail in the following. The possibility for the existence of the fundamental wave will appear with still more evidence from the general examination of the flow around an aerofoil, to which we shall now pass.

The same scheme which we have developed for a symmetrical body can be applied to an asymmetrical body like an aerofoil. All that has been said relative a symmetrical body has to

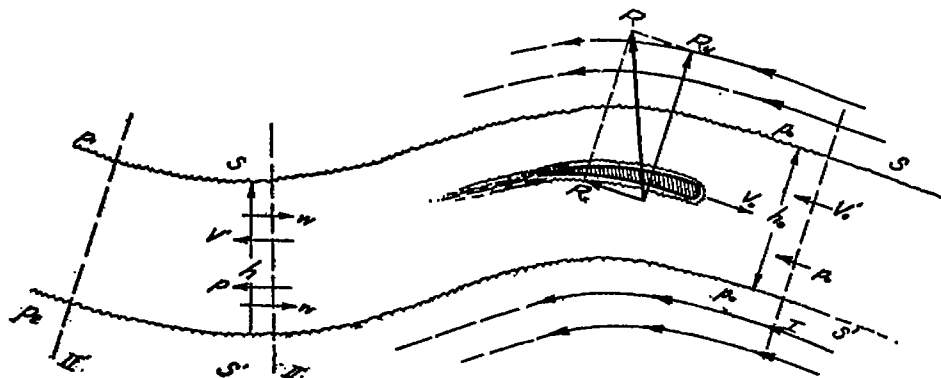


FIG. 27.

be directly transferred to the aerofoil. The difference will consist in the fact that, as the aerofoil has a lift component due to the fluid, there must be a fluid momentum corresponding to that lift. That is to say, the relative flow behind the aerofoil must be deflected downward. The schematical flow around an aerofoil is represented in figure 27. Let us now imagine for one moment the aerofoil immobile and the stream running on it with the velocity V_0 . The fluid velocity outside the stream boundaries SS and $S'S'$ will also be V_0 . In such a condition it is easy to see that there will be a tendency to straighten the deflected stream by the outside stream. If we assume the possibility for the stream between the boundaries SS and $S'S'$ to become horizontal after section II, the application of the momentum theorem for the lift,

between two sections such as I and II, will show an increased pressure p_2 above and a decreased pressure p_1 below. In such a condition it would be difficult to imagine how the theorem of moments of momentum applied to the stream portion between SS and $S'S'$ on one side and I and II on the other side could be satisfied. Such a flow appears impossible, and it is easy to see that, after its downward deflection, the stream, by the difference of pressure ($p_2 - p_1$) must necessarily be deflected upwards. It thus becomes evident that behind an aerofoil we have a wave motion of the relative stream. The instability of the vortex boundary sheets SS and $S'S'$ also lead to this conclusion. The wave motion which is to be expected is represented schematically in figure 28. It is sufficient to look at this last figure to see at once that the wave motion obtained is governed by a system of vortices in quincunx, rotating in one sense for the upper row and in an inverse sense for the lower row. We thus see that behind the aerofoil we can expect to see the phenomenon of the *fundamental wave* mentioned in the foregoing.

The phenomenon of the secondary wave can also take the place for the aerofoil.

We are thus brought to the conclusion that a simple deflection of the relative stream behind the aerofoil is not to be expected. Nevertheless, the preliminary study of the stream deflection behind an aerofoil is of interest for the following reason.

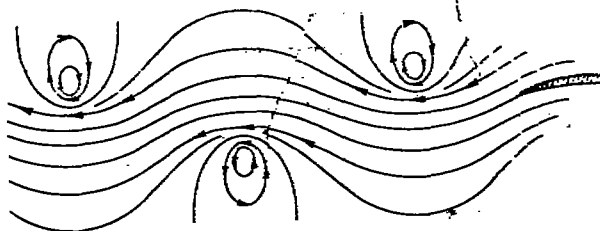


FIG. 28.

Let us consider an aerofoil II disposed in the wake of another I. This aerofoil II will then be submitted to a periodical stream. Let us assume for simplicity that both the magnitude of the velocity V of the flow running on the aerofoil II and the angle of attack i vary according to sinusoidal laws, so that as a first approximation we consider

$$V = V_0 + v \sin \frac{2\pi}{T} t; \quad i = i_0 - j \sin \frac{2\pi}{T} t$$

T being the period, v and j the amplitudes of variation of V and i , t the variable time. The difference of sign in the above expression denotes the fact that the velocity V is assumed increasing when the angle of attack decreases, and vice versa. We shall also assume that for the instantaneous values of V and i the resultant air-resistance R of the aerofoil II can be expressed by the formula:

$$R = KA V^2 i = \lambda V^2 i$$

writing $KA = \lambda$. Let us now, under these assumptions, calculate the mean value R_m of R . We have

$$\begin{aligned} R_m &= \frac{1}{T} \int_0^T R dt \\ &= \frac{1}{T} \int_0^T \lambda \left(V_0 + v \sin \frac{2\pi}{T} t \right)^2 \left(i_0 - j \sin \frac{2\pi}{T} t \right) dt \\ &= \frac{\lambda V_0^2 i_0}{T} \int_0^T dt + \frac{\lambda V_0^2 (2vi_0 - jV_0)}{T} \int_0^T \sin \frac{2\pi}{T} t dt + \\ &\quad + \frac{\lambda v (vi_0 - 2jV_0)}{T} \int_0^T \sin^2 \frac{2\pi}{T} t dt - \frac{\lambda jv^2}{T} \int_0^T \sin^3 \frac{2\pi}{T} t dt. \end{aligned}$$

As is well known, the integrals of the odd exponents of the Sine are equal to zero, and

$$\frac{1}{T} \int_0^T \sin^2 \frac{2\pi}{T} t = \frac{1}{2}$$

So that we finally get

$$R_m = \lambda V_o^2 \left(i_o - \frac{v}{V_o} \right) + \frac{1}{2} \lambda v^2 i_o$$

For small values of v the term $\frac{1}{2} \lambda v^2 i_o$ can be neglected, and thus

$$R_m \cong \lambda V_o^2 \left(i_o - \frac{v}{V_o} \right)$$

and we are brought to the following conclusion: It is easy to represent such a periodical flow that an aerofoil disposed in it will show, for the mean value of the air resistance, an apparent decrease of the angle of attack. The aerofoil will thus appear as if placed in a downward deflected stream. The study of the apparent stream deflection behind the aerofoil is thus justified.

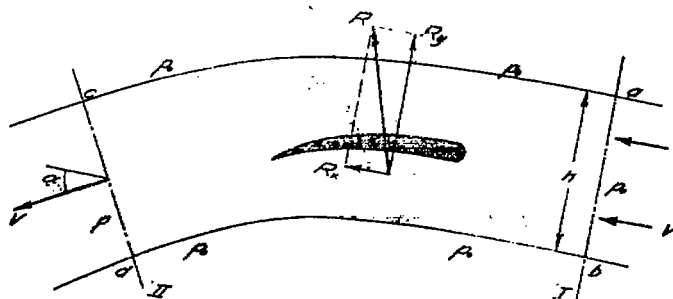


FIG. 29.

The apparent stream deflection.—If we make the assumption that all the lift of an aerofoil is due to the momentum created by the deflection of the stream downward, the angle of deflection of the stream can be easily calculated. This calculation will also give us a mean value of the height of the stream disturbed by the presence of an aerofoil in the fluid.

Let us consider a unit of length of an aerofoil and draw around it a contour a, b, c, d defined as follows (see fig. 29):

The side ab of the contour is a plane cross section of the stream taken at such a distance before the aerofoil that the flow in that section is not disturbed by the presence of the aerofoil. It is the cross section I of the relative stream. The two sides ac and bd of our contour are taken along two stream lines at such a distance that the local phenomena created by the presence of the aerofoil in the fluid do not reach them, and that the pressure on these two stream lines is equal to the outside pressure. At the end, the side cd , constituting the section II of the relative stream, is a plane cross section taken at such a place that the velocity of the stream has nearly taken its original value and the pressure has also nearly reached its original value. These last assumptions are only a certain approximation. Let us now calculate the increments of the components of the momentum of the fluid running out of this contour for a unit of length of the aerofoil, along the velocity and along the normal to the velocity.

It is easy to see that the increment of the components of the fluid momentum along the velocity is equal to

$$h\delta V^2 - h\delta V^2 \cos \alpha = h\delta V^2 (1 - \cos \alpha)$$

and that the increment of the components of the fluid momentum along the normal to the velocity is equal to

$$h\delta V^2 \sin \alpha$$

As these two components of the momentum must be equal to the two components of the fluid pressure on our aerofoil, we must have

$$(18) \quad R_x = k_x \delta b V^2 = h \delta V^2 (1 - \cos \alpha)$$

$$(19) \quad R_y = k_y \delta b V^2 = h \delta V^2 \sin \alpha$$

where b is the breadth of the aerofoil and δ the air mass density. From the last relations it follows that

$$k_x b = h(1 - \cos \alpha) \quad ; \quad k_y b = h \sin \alpha$$

$$\frac{k_x}{k_y} = \frac{1 - \cos \alpha}{\sin \alpha} = \operatorname{tg} \frac{\alpha}{2}$$

but, as we have

$$\sin \alpha = \frac{2 \operatorname{tg} \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}}$$

we easily get the value of

$$(20) \quad \sin \alpha = \frac{2 \frac{k_x}{k_y}}{1 + \frac{k_x^2}{k_y^2}} = \frac{2 k_x k_y}{k_x^2 + k_y^2}$$

Having the value of $\sin \alpha$ we easily get the value of the stream height disturbed by the aerofoil:

$$(21) \quad h = \frac{k_y b}{\sin \alpha} = \frac{b k_y^2 \left(1 + \frac{k_x^2}{k_y^2}\right)}{2 k_x} = \frac{b (k_x^2 + k_y^2)}{2 k_x}$$

For small values of the angle α and negligible values of k_x^2 , we can write

$$(22) \quad \alpha \approx \frac{2 k_x}{k_y}$$

$$(23) \quad h \approx \frac{b k_y^2}{2 k_x}$$

The value of these formulas, the deduction of which is only based on certain assumptions, lies in the fact that they indicate from which quantity depends the apparent angle of deflection of the stream and the height of the fluid stream disturbed by the presence of the aerofoil. Further, these formulas give values for both quantities of the order of magnitude as obtained from experiment.¹

The formula (22) is capable of an interesting geometrical interpretation. If we draw a plane normal to the resultant air resistance R , we then see that the stream is, so to speak, reflected on this plane. (See fig. 30.)

From these very simple considerations we see that the conceptions developed in turbine theory, where it has been assumed that the fluid runs off from a turbine wing in the direction of the tangent to its trailing edge, are absolutely inadmissible. The apparent direction of the stream behind an aerofoil or a turbine wing depends not only upon the direction of the tangent

¹ For example, let us take (the units used being m., kg., and sec.)

$$\begin{aligned} i &= 6^\circ \\ k_y &= \frac{k_x^2}{b} = \frac{K i}{b} = \frac{1/200 \cdot 6}{1} \approx 0.003 \\ k_x/k_y &= 1/16 \end{aligned}$$

we then have

$$h = \frac{b k_y^2}{2 k_x} = \frac{b k_y}{2 \frac{k_x}{k_y}} = \frac{0.1/4}{2 \cdot 1/16} = 0.8$$

a quantity which is of the order of what experience with biplanes indicates to be negligible influence of the mutual interference of the wings.

to the trailing edge but upon the form of all the parts of the aerofoil or the wing. The general results to which we are brought in this elementary calculation are certainly only of a first approximation, but they give a rational description of the general phenomena.

We shall now proceed to examine the problem of the apparent stream deflection behind the aerofoil to a second approximation.

When the stream meets the aerofoil, as the result of the impact which takes place there must be a certain amount of energy dissipated inside the fluid. Let us designate for the section I before the aerofoil, by p_0 , V_0 and h_0 , the pressure, the flow velocity and the relative stream height disturbed by the aerofoil, and by p , V and h , the values of these same quantities for the section II behind the aerofoil. We shall apply to the fluid between the sections I and II the momentum theorem taking account of the dissipation of energy by two limiting assumptions. The first assumption will consist in considering in the section II the velocity $V = V_0$; the second, in admitting $p = p_0$. Under such conditions the dissipative force F will be equal to the drag of the aerofoil for an angle of attack equal to zero, as follows from the foregoing. For other values of the angle of attack, the coefficient f has to be considered, for a given aerofoil, and as a function of the angle of attack. In the case of $V = V_0$, as we have $F = h(p_0 - p)$, the dissipative force F can be conceived as applied in the section II normally to that section.

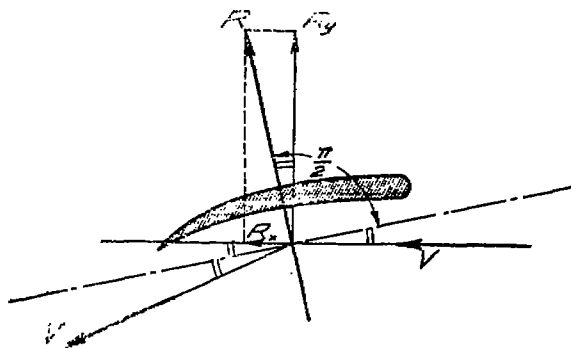


FIG. 30.

Applying the momentum theorem, in the case of the first assumption, it is easy to see that we have: (See fig. 27, and compare with the similar equations (18) and (19)).

$$(24) \quad R_x = k_x \delta b V^2 = h \delta V^2 - h \delta V^2 \cos \alpha + f \delta b V^2 \cos \alpha$$

$$(25) \quad R_y = k_y \delta b V^2 = h \delta V^2 \sin \alpha - f \delta b V^2 \sin \alpha$$

which equations express the fact that the drag and lift are equal to the corresponding components of the variation of the fluid momentum to which are added the components of the dissipative force F .

Dividing the last equations by $\delta b V^2$ we get

$$(26) \quad k_x = h/b - h/b \cos \alpha + f \cos \alpha$$

$$(27) \quad k_y = h/b - f \sin \alpha$$

from which follows

$$\sin \alpha = \frac{k_y}{h/b - f} \quad \text{and} \quad \cos \alpha = \frac{h/b - k_x}{h/b - f};$$

and, since $\sin^2 \alpha + \cos^2 \alpha = 1$, we have

$$(h/b - f)^2 = k_y^2 + (h/b - k_x)^2$$

or, removing the parentheses and multiplying, we finally find

$$(28) \quad h/b = \frac{k_y^2 + k_x^2 - f^2}{2(k_x - f)}$$

or

$$(29) \quad h = \frac{b(k_y^2 + k_x^2 - f^2)}{2(k_x - f)}$$

By aid of the relation (28) we find

$$h/b - f = \frac{k_y^2 + (k_x - f)^2}{2(k_x - f)}$$

and consequently

$$(30) \quad \sin \alpha = \frac{2k_y(k_x - f)}{k_y^2 + (k_x - f)^2}$$

Neglecting the squares of k_x and f which are of the same order and very small quantities for actual aerofoils and small angles of attack, we get

$$(31) \quad h \approx \frac{bk_y^2}{2(k_x - f)}$$

$$(32) \quad \sin \alpha \approx \alpha \approx \frac{2(k_x - f)}{k_y}$$

In the case of the second assumption, that is, for $p = p_0$ and $\Delta C = \delta/2(V_0^2 - V^2)$, we have (see fig. 27 and compare with the similar equations (18) and (19))

$$(33) \quad k_x \delta b V_0^2 = h_0 \delta V_0^2 - h \delta V^2 \cos \alpha$$

$$(34) \quad k_y \delta b V_0^2 = h \delta V^2 \sin \alpha$$

or

$$(35) \quad k_x = h_0/b - h/b \frac{V^2}{V_0^2} \cos \alpha$$

$$(36) \quad k_y = h/b \frac{V^2}{V_0^2} \sin \alpha$$

and taking account of the condition of continuity $h_0 V_0 = h V$ we get

$$(37) \quad h_0/b - k_x = h_0/b \frac{V}{V_0} \cos \alpha$$

$$(38) \quad k_y = h_0/b \frac{V}{V_0} \sin \alpha$$

Taking into account the relations (8), (15), and (16), we find

$$1 - \frac{V^2}{V_0^2} = \frac{2F}{\delta h_0 V_0^2} = \frac{2fb}{h_0};$$

so that the ratio V^2/V_0^2 is seen to be equal to

$$(39) \quad \frac{V^2}{V_0^2} = 1 - 2f \frac{b}{h_0}$$

Squaring the relations (37) and (38), adding them and substituting in the last the foregoing value of V^2/V_0^2 , we find

$$(40) \quad (h_0/b - k_x)^2 + k_y^2 = (h_0/b)^2 (1 - 2fb/h_0);$$

from which relation we directly find

$$(41) \quad h_0/b = \frac{k_x^2 + k_y^2}{2(k_x - f)}$$

Introducing this last value of h_0/b in the equation (38) and substituting in it for V/V_0 its value we find

$$(42) \quad \sin \alpha = \frac{2k_y(k_x - f)}{\sqrt{(k_x^2 + k_y^2)} [k_x^2 + k_y^2 - 2f \cdot 2(k_x - f)]}$$

Finally, neglecting for reasons already mentioned the squares of k_x and f , we find

$$(31) \quad h_0 \approx \frac{bk_y^2}{2(k_x - f)}$$

$$(32) \quad \sin \alpha \approx \alpha \approx \frac{2(k_x - f)}{k_y}$$

If we now introduce these last values of h_0 and $\sin \alpha$ in the equation (38), we at once see that

$$(43) \quad V \cong V_0 \text{ and consequently } h \cong h_0.$$

This shows that the difference between h and h_0 , or between V and V_0 , is only of the order of k_x^2 ; that is, of the second order compared with k_x , and consequently negligible. We thus see that in the wake of the aerofoil we must have nearly the same velocity as in front, but a decreased pressure.

All the foregoing discussion brings us to the following important conclusions. Both assumptions for negligible values of k_x^2 and f^2 bring us to the same mean values of the height h_0 of the stream disturbed by the aerofoil and the angle α of the apparent stream deflection, the variation in the magnitude of the stream velocity and stream cross section being of the order of k_x^2 , and therefore negligible. The expressions (31) and (32), compared with the corresponding expressions (22) and (23), give for the height h larger values and for the angle α smaller values.

Let us now see how far the results of the foregoing discussion are verified by experiment.

In his "*Nouvelles recherches sur la resistance de l'air et l'aviation*," G. Eiffel, on pages 165-170, gives the results of measurement of the air resistance of an aerofoil disposed in the wake of another. Let us designate by I the aerofoil disposed directly in the wind stream, and by II the aerofoil disposed in the wake of I. In figure 31, Z_I is the zero line of the first aerofoil and

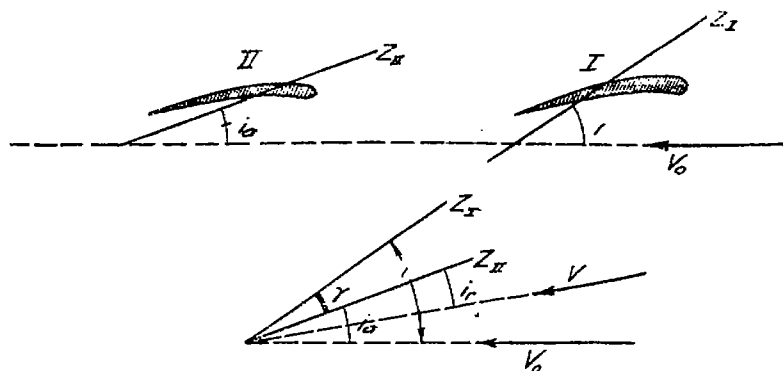


FIG. 31

Z_{II} the zero line of the second, V_0 , the velocity of the stream before the first aerofoil, and i_a and i_r the angles of attack of the first and second aerofoils relative to the velocity V_0 . Eiffel shows that for the aerofoil II the air resistance is such that the angle of attack of that aerofoil, instead of being equal to i_a , appears to be reduced to the value i_r , smaller than i_a . Eiffel calls i_a the "apparent" incidence and i_r the "real" incidence. Eiffel gives the corresponding values of i_a and i_r for different angles γ between the zero planes of the two aerofoils; and these we reproduce in the Table A, only referring all the angles to the corresponding zero lines.

TABLE A.

i	$\gamma=2^\circ$			$\gamma=4^\circ$			$\gamma=6^\circ$		
	i_a	i_r	α	i_a	i_r	α	i_a	i_r	α
5°	3°	1° 4'	3° 5'	5°	-0,4	3° 4'	5°	-1° 6'	4° 6'
7	5	2,7	4,8	5	0,6	4,4	5	-0,4	5,4
9	7	3,9	5,1	7	1,9	5,1	5	0,8	6,2
11	9			9	3,1	5,9	7	2,1	5,9
13	11			11	4,5	6,5	9	3,2	7,8
15	13			13	5,6	7,4	11		
17									

Let us now conceive the stream velocity behind the first aerofoil as deflected and having the general direction V . The angle of attack, i_r , is the angle between V and Z_{II} , so that, if α is the value of the stream deflection angle, we must have (see fig. 31)

$$i_a - i_r = \alpha$$

In the Table A, by aid of the values of i_a and i_r , the values of α are calculated for different values of the angles of attack i of the aerofoil I, and three different values of the angle γ , 2° , 4° , and 6° , respectively. By using the values obtained, in figure 32 are plotted the curves of $\alpha = f(i)$ for the three values of $\gamma = 2^\circ$, 4° , 6° . It is easy to see that all the plotted points define well enough a curve, which means that behind the aerofoil I, we really have an apparent stream deflection depending only upon the aerofoil I, and its angle of attack, i . The angle of mutual inclination of the two aerofoils has no influence on that phenomenon, as ought to be expected. The aerofoil, II, has simply to be considered as disposed in a stream deflected by the aerofoil I, whose velocity in magnitude is nearly the same as in front of the aerofoil I. Knowing the relation between α and i , we can easily calculate the air resistance of any other aerofoil, as the aerofoil II, disposed behind the aerofoil I.

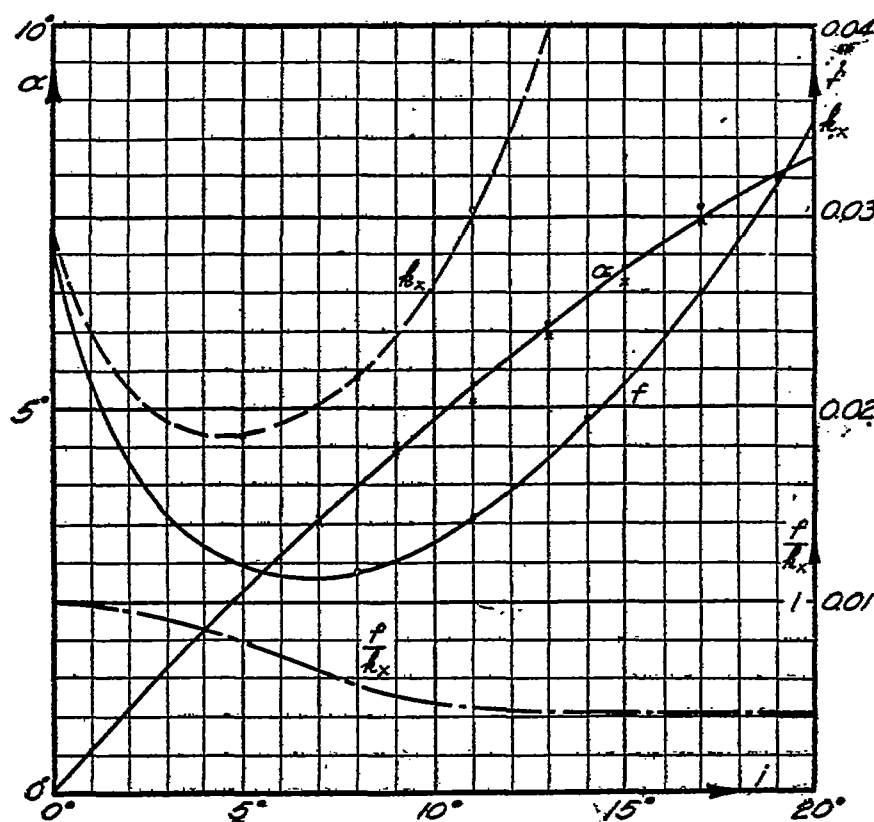


FIG. 32.

Let us now calculate according to these last data the values of the coefficient f . This calculation is made by aid of the formula (32), in the Table B, where are reproduced all the data concerning the aerofoil I, which are necessary for that calculation.

TABLE B.

i	α°	α radians	k_γ	k_x	$\frac{\alpha k_\gamma}{2}$	$f = k_x - \frac{\alpha k_\gamma}{2}$	f/k_x
5	2,5	0,044	0,176	0,0182	0,00388	0,0144	0,79
8	4,0	0,070	0,268	0,0217	0,0101	0,0116	0,585
11	5,3	0,093	0,383	0,0308	0,0178	0,013	0,42
14	6,5	0,133	0,475	0,0454	0,0288	0,0196	0,41
17	7,5	0,130	0,563	0,062	0,0366	0,0254	0,41

The values of the coefficient f obtained are plotted in figure 32, where, for comparison, is also plotted the curve of k_x . In the last column of the Table B is also calculated the ratio of f to k_x . These last values are very suggestive. They show that the dissipative force F

decreases when the angle of attack increases, and that for mean values of the last it is equal to nearly one-half of the drag.

Finally I will remark that the energy dissipation, of which account was taken in the foregoing, and to which corresponds the dissipative force F , is that dissipation of energy which takes place in the immediate neighborhood of the aerofoil. It is evident that all the energy spent to move an aerofoil in air is dissipated in the surrounding medium; but one part is dissipated in the direct neighborhood of the aerofoil and corresponds to skin friction and turbulence connected with it, which we evaluate by FV_0 ; and another part is dissipated by the oscillatory motion of the air left behind the aerofoil—that is, by the damping through viscosity of the fundamental and secondary waves created.

The experimental study of the apparent stream deflection behind the aerofoil in the light of the ideas here developed is very important for many problems connected with the design of airplanes, propeller, and turbines. It must, however, be remembered that we have to do only with an *apparent* stream deflection, the real motion of the fluid behind an aerofoil or turbine wing being generally periodical.

Short review of some propositions on vortices.—Before proceeding to examine the question of the tip vortices and the fundamental and secondary wave, I shall state briefly some well-known propositions on vortices in general.

Let us consider a small circle of radius r rotating in its plane with an angular velocity ω . Each part of the contour of this circle has a velocity equal to $r\omega$. If we now calculate for the contour of the circle, the quantity which in hydrodynamics is called the circulation—that is, the integral of the velocity v along the contour of the circle—we find (see fig. 33)

$$(44) \quad I = \int dI = \int_0^{2\pi} r\omega r d\varphi = 2\pi r^2 \omega = 2\omega d\sigma$$

or

$$(45) \quad \omega = \frac{I}{2d\sigma}$$

where I is the circulation along the contour of the circle and $d\sigma$ the area of the small circle. We therefore see that the angular velocity ω of a rotating circle is equal to the circulation I divided by the double of the surface of the circle.

Let us now consider a fluid element having the velocity (u, v, w) at a point (x, y, z) in a moving fluid mass. Let us draw through this point the axes X, Y, Z parallel to a system of triorthogonal immobile axes and calculate the circulation along an elementary contour with sides equal to dx and dy , as shown in figure 34. We have

$$dI = udx + \left(v + \frac{\partial v}{\partial x} dx\right) dy - \left(u + \frac{\partial u}{\partial y} dy\right) dx - vdy = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx dy$$

The quantity

$$(46) \quad \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{dI}{2dxdy} = \omega_z$$

which we designate by ω_z is called the component of the vortex with reference to the Z axis. (Compare formula (45).) In a similar manner

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right); \quad \omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

are the vortex components with reference to the X and Y axes. We have

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$$

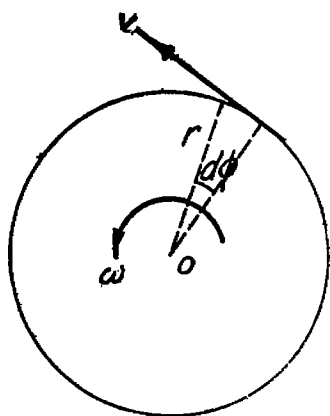


FIG. 33.

The doubles of the vortex components $2\omega_x$, $2\omega_y$, $2\omega_z$ are the determinants of the matrix.

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \\ 2\omega_x & 2\omega_y & 2\omega_z \end{vmatrix}$$

As a consequence of continuity, the following proposition holds for vortices:

If we have a vortex motion at one point of a fluid, we must necessarily have vortex motion at all the points of a line going through that point. Such a line of small cross section $d\sigma$ at each point of which the vortex has a finite value is called a *vortex filament*. The quantity

$$(47) \quad 2\omega d\sigma = I$$

is called the *intensity* of the vortex filament. A vortex filament can never begin or end in a fluid. It must be a closed contour or have its ends on the boundary surface of the fluid. The cross section of a vortex filament can be variable, but its intensity is always constant along the whole filament—that is to say, for a vortex filament, we have

$$2\omega d\sigma = \text{Const.}$$

The vortex filament is always constituted of the same fluid particles—that is, the vortex filament moves with the fluid. A system of vortex filaments disposed close together form a vortex tube.

Let us consider a fluid mass in motion with a vortex filament in it, and let us draw a surface across the fluid and take a contour on that surface. The circulation along the contour is equal to twice the sum of the elements of the surface multiplied by the components, along the corresponding normal to the surface, of the vortices on that surface. (See fig. 35.)

$$(48) \quad I = \int V ds = 2 \iint \omega_n d\sigma$$

where ω_n is the vortex component normal to the considered surface. This last relation constitutes Stokes's theorem. In the application of this theorem two cases have to be distinguished. If the contour, by progressive shrinking, can be reduced to a point without leaving the fluid, the space occupied by the fluid is said to be "simply connected." If, inside the space occupied by the fluid, we have solid bodies or holes crossing the fluid mass, not every contour in the fluid can be reduced, by shrinking, to a point; and the space containing the fluid is said to be "not simply connected." In a simply connected space, if the circulation along the contour has a finite value it means that vortices are crossing the inside of the contour and the double of the sum of the components of the vortices normal to a surface containing that contour multiplied by the corresponding elements of the surface is equal to the circulation along that contour. In a not simply connected space a finite value of the circulation can also mean that solid bodies or holes are crossing the inside of the contour.

In a fluid mass in motion with vortices in it, the velocity at each point depends upon the distribution of the vortices. Each element of each vortex contributes to the velocity at each point. The components of the velocity at a point due to a vortex element ds of a vortex filament of intensity I is equal to (see fig. 36)

$$(49) \quad d\vec{u} = \frac{I ds \sin \varphi}{4\pi r^2}$$

where r is the distance between the point considered and the vortex element, and φ is the angle between ds and r . The direction of the velocity $d\vec{u}$ is normal to the plane containing r and ds and has the sense of the rotation around ds in the sense of the vortex. The velocity at a point is the geometrical sum of the velocity components due to all the elements of the vortices contained in the fluid.

In the case of one single rectilinear vortex filament in an infinite fluid mass, by reason of symmetry the velocity is the same for all the points at the same distance from the vortex. If

we therefore calculate the circulation along a circle contained in a plane normal to the vortex, and having its center on the vortex, we get

$$I = 2\omega d\sigma = 2\pi r v$$

so that

$$(50) \quad v = \frac{I}{2\pi r}$$

That is to say, the fluid velocity for points around a straight vortex filament is equal to the intensity I of the vortex divided by 2π times the distance r from the point considered to the vortex. The vortex itself does not move.

Let us consider now the case of two straight parallel and infinite vortices of equal intensity I rotating in inverse senses and contained in an infinite fluid mass. In figure 37 the vortices are represented in cross section and in plane, and L is the distance between the vortices. We will refer the vortices to a system of triorthogonal axes X, Y, Z as shown in figure 37, the origin being in the middle between the vortices and the Z axis being perpendicular to the plane containing the vortices.

Let us first calculate the velocity at the point A , at the distance x from the origin O , due to one of the vortices. Applying the foregoing formula we have

$$u = \int_{-\infty}^{+\infty} \frac{I dy \sin \varphi}{4\pi r^2}$$

and from figure (31) we easily see that we have

$$y = r \cos \varphi; (L/2 - x) = r \sin \varphi$$

from which follows

$$\frac{y}{L/2 - x} = \cotg \varphi$$

and

$$\frac{dy}{L/2 - x} = \frac{-d\varphi}{\sin^2 \varphi}$$

Substituting these last in the foregoing integral we get

$$u = \frac{I}{4\pi(L/2 - x)} \int_{-\pi/2}^{\pi/2} -\sin \varphi \, d\varphi$$

and integrating we finally have

$$u = \frac{I}{4\pi(L/2 - x)} \cos \varphi \Big|_{-\pi/2}^{\pi/2} = \frac{I}{2\pi(L/2 - x)}$$

Considering x equal to $L/2$ we will find the velocity of one of the vortices produced by the other which is equal to

$$(51) \quad u = \frac{I}{2\pi L}$$

This is the value of the velocity with which both vortices move parallel to the Z axis. For the velocity between the two vortices produced by both we will have

$$(52) \quad v = \frac{I}{2\pi(L/2 - x)} + \frac{I}{2\pi(L/2 + x)} = \frac{IL}{2\pi(L^2/4 - x^2)}$$

For points in the middle between the vortices the velocity is equal to

$$(53) \quad v = \frac{2I}{\pi L} = 4u$$

For other points between the vortices the velocity follows a parabolic law, and, as a first approximation, can be considered as nearly uniform in the middle part between the vortices when the cross sections of the vortices are small relative to the distance between them. (See fig. 37.) We therefore see that the velocity of the fluid in the middle between the vortices is exactly equal to four times the velocity of the vortices themselves and that the velocity of the fluid between the vortices relative to them is equal to twice the velocity of the vortices themselves.

We have now all the necessary references for the following:¹

The tip vortices.—Let us consider an aerofoil moving with a constant and uniform velocity V in a fluid mass and let us designate by dz an element of length of the aerofoil, z being the distance of the element of the aerofoil considered from its middle cross section. (See fig. 38.) Let dR_y be the lift corresponding to the element of the aerofoil considered. The quantity

$$\frac{dR_y}{dz}$$

is the lift per unit of length at the distance z from the middle cross section of the aerofoil. According to Kutta's theorem (see Note I) we must have

$$(54) \quad \frac{dR_y}{dz} = \delta V I$$

so that

$$(55) \quad I = \frac{dR_y}{\delta V dz}$$

is the value of the circulation along such a contour as I , embracing the aerofoil at the cross section considered. (See fig. 38.) When z varies, and until we do not approach too near to the tips of the aerofoil, the value of I is nearly constant. As a first approximation for the mean value of I we can take

$$(56) \quad I = \frac{R_y}{\delta V L}$$

where L is the length of the aerofoil. Let us now move our contour to the tips of the aerofoil, and just before the value of I begins to change we let the contour follow the fluid in its motion. According to a theorem of William Thomson (Lord Kelvin), the circulation along the contour moving with the fluid must be invariable, so that when the contour reaches such a position as II (see fig. 38) the circulation will have the same value as in the position I. But in the position II we have no solid body inside the contour, and consequently we must, according to Stokes's theorem, have a vortex traversing that contour.

We so come to the conclusion that vortices must necessarily run off from the tips of aerofoils, and that the mean value of the intensity of this vortex must be

$$(57) \quad I = \frac{R_y}{\delta V L} = k_y b V$$

According to this statement, the tip vortices disappear only when the lift vanishes; that is to say, when the relative wind blows on the aerofoil along the zero plane.

Having established the value of the intensity of the tip vortices, let us now consider their influence on the air resistance of the aerofoil. For the simplicity of the explanation we will

¹ For more detailed references on vortices see the classical treatise on mechanics by *Paul Appell*, "Traité de mécanique rationnelle," Tome III, Chap. XXXV, p. 389, and p. 478, § 791, and also "Aeronautics in Theory and Experiment," by W. L. Cowley and H. Levy, Chap. III.

put ourselves in the simplest case and will assume that the tip vortices are of small cross section compared with the length of the aerofoil, so that they can be considered as vortex filaments.

If we follow an aerofoil moving with a constant velocity V , we see that the tip vortices run parallel to themselves, as should be the case for parallel vortices rotating in inverse senses. The velocity with which these two vortices displace themselves in the stream, at a distance

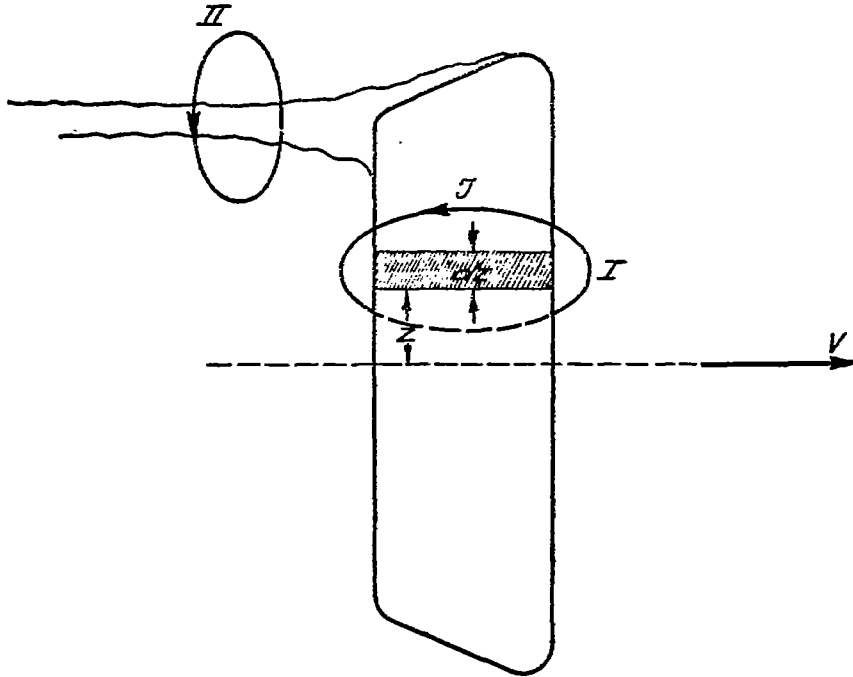


FIG. 38.

from the aerofoil where the influence of the last can already be neglected, is, according to the foregoing, equal to

$$(58) \quad u = \frac{I}{2\pi L}$$

and is normal to the plane containing the vortices. The motion of the fluid between the two vortices in the middle part relative to the vortices is equal to

$$(59) \quad 2u = \frac{I}{\pi L}$$

So that the mean value of the flow velocity between the tip vortices relative to the aerofoil is equal to (see fig. 39)

$$\bar{V} + 2\bar{u}$$

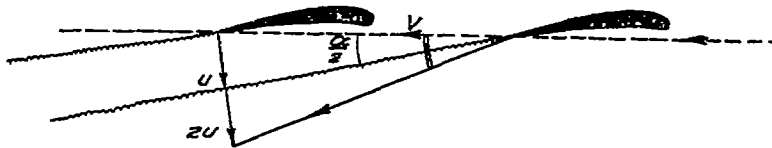


FIG. 39.

We therefore see that as a first approximation we can consider the tip vortices as bisecting the angle which the mean value of flow velocity between them makes with the original direction of the stream running on the aerofoil (see fig. 39). We also see that the angle of deflection of the tip vortices downward is equal to

$$(60) \quad \sin \frac{\alpha}{2} = \frac{u}{V}$$

Substituting the value of the velocity u by its expression (58), and in the last the circulation I by its value (57) we find

$$(61) \quad \sin \frac{\alpha}{2} = \frac{k_y b}{2\pi L}$$

If there were no other circumstances producing the deflection of the stream downward, this would be the correct mean value of the deflection angle.

If now, for the calculation of the part of the drag due to the tip vortices, we should apply the momentum theorem, assuming that the mean value of the velocity behind the aerofoil is as a first approximation the resultant of the velocity V of the aerofoil and the velocity $2u = I/\pi L$, we would find, according to the calculation already made on page 34, that

$$\tan \frac{\alpha}{2} = \frac{k_x}{k_y} = \frac{\sin \frac{\alpha}{2}}{\sqrt{1 - \sin^2 \frac{\alpha}{2}}}$$

and substituting in the last expression the value of $\sin \frac{\alpha}{2}$ given by (61) we would get

$$\frac{k_x}{k_y} = \frac{s k_y}{\sqrt{1 - s^2 k_y^2}}$$

where by s we have designated the quantity

$$s = \frac{b}{2\pi L}$$

or, finally,

$$(62) \quad k_x = \frac{s k_y^2}{\sqrt{1 - s^2 k_y^2}}$$

which expression gives us a first approximation to the value of the drag due to the tip vortices. If we compare the values which the last formula gives for the drag with those values which direct experience shows for actual aerofoils, it will be easy to see that the drag of an aerofoil is much larger than that calculated by the formula (62).¹ We afterwards see that the value of the drag given by the formula (62) decreases with the aspect ratio of the aerofoil and is equal to zero for an infinite value of the aspect ratio. These last conclusions are very important. They first show us that that part of the drag of an aerofoil which is due to the tip vortices practically disappears for a value of the aspect ratio greater than a certain one. We afterwards see that the most important part of the drag, practically the whole, for a sufficient value of the aspect ratio, is due to other circumstances than the tip vortices, which circumstances are, as we already know, on one hand the energy dissipation in the direct neighborhood of the aerofoil—that is, skin friction and turbulence—and on the other, the fundamental and secondary waves.

We can now perfectly conceive the importance of the phenomena which take place on the surface of contact of the solid and fluid bodies. We can not expect to be able to calculate the drag of an aerofoil before these phenomena are fully understood, as has been already stated in the preamble.

¹ The easiest way to see it is the following:

Remarking that in the formula (62) $s^2 k_y^2$ is a very small quantity we can write

$$k_x/k_y = s k_y = \frac{b k_y}{2\pi L} = \frac{b}{2\pi L s} \cdot K$$

For actual aerofoils, as mean values we have

$$b/L = 1/6; \quad \frac{b}{2\pi L s} \approx 0.2; \quad K = 1/200$$

So that

$$k_x/k_y \approx \frac{1}{1000}$$

Which is out of proportion with any observed value for the drag-lift ratio.

All this discussion brings us to very important conclusions. We have fully understood the influence of the aspect ratio on the drag and, so to say, the mechanism of its influence. When the aspect ratio increases, the part of the drag due to the tip vortices practically disappears, the influence of the tip vortices becoming negligible. The last fact taking place, the shape of the tip vortices is also negligible. It follows that the shape of the tips of the aerofoils has a negligible influence on the air resistance of the aerofoil, if only the aspect ratio has a sufficient value.¹

We can now formulate the two following propositions:

1. *The tip vortices have an influence on the drag of the aerofoil; but this decreases with the increase of the aspect ratio and practically disappears for a certain value of the last.*
2. *When the aspect ratio of an aerofoil has a sufficient value, the influence of the form of the tips of the aerofoil on the air resistance is negligible.*

These two last conclusions are in full agreement with all experience with aerofoils up to this time.

We have been able, by the analysis of the tip vortices phenomenon, to understand the reasons which require for the aerofoils a certain value of the aspect ratio and have deduced the slight influence of the tip forms when the aspect ratio has a sufficient value.

Short Review of the Properties of Systems of Parallel Vortex Rows.—We will call *vortex row*, a system of an infinite number of infinite rectilinear parallel and equidistant vortices of infinitely small cross section and of strength I equal in magnitude and sense, disposed on one straight line. We will give a short review of the properties of *systems of parallel vortex rows*, which play a very important rôle in the phenomenon of fluid resistance.

The fluid mass in which the infinite parallel rectilinear vortices are considered is assumed of infinite dimensions in all senses, having in infinity a velocity equal to zero, or moving as a whole with a velocity constant in magnitude and direction, independent of the motion which can take place inside the fluid.

We consider the whole system of vortices cut by a plane normal to them and their mutual positions defined by the positions of their sections in that plane.

We will first consider in their general outlines the conditions which must be satisfied by a system of infinite parallel and rectilinear vortices in order to maintain an invariable configuration.

In figure 40 are represented the sections of a system of vortices, as above mentioned, cut by a plane normal to them.

The vortices are assumed to maintain for one moment an invariable configuration and are referred to the orthogonal (X , Y) axes moving with the vortex system. Let us concentrate our attention on one of the vortices of the system, say A , with the coordinates x and y , and consider this vortex A undergoing an infinitely small displacement δx and δy relative to the other vortices, the last maintaining their configuration, and let u and v be the components of the velocity of the vortex A . As is well known; u and v are the components of the velocity produced at the point x , y by the other vortices of the system. The displacement of the vortex A will produce the variations du and dv of its velocity components, which will represent the components of the velocity of the vortex A relative to the other vortices, and we will have:

$$du = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y$$

$$dv = \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y$$

The vortex A will be stable for every virtual displacement $(\delta x, \delta y)$ provided du and dv , respec-

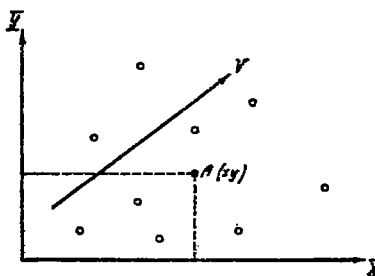


FIG. 40.

¹ And if the tips do not have any extravagant form which could give rise to additional local air resistance.

tively have signs opposite to the signs of the displacements δx and δy . But it is easy to see that, so long as

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

have finite values, it is always possible to imagine a displacement $(\delta x, \delta y)$ for which du and dv will have the same signs as δx and δy . In other words, a stable displacement of a single rectilinear vortex among others maintaining their configuration appears as impossible.

But in the case of

$$(63) \quad \frac{\partial u}{\partial x} = 0; \frac{\partial u}{\partial y} = 0; \frac{\partial v}{\partial x} = 0; \frac{\partial v}{\partial y} = 0$$

we have

$$du = 0; dv = 0$$

for all infinitely small virtual displacements of a single vortex. Under such conditions the vortex considered will be in a state of neutral relative equilibrium among the other vortices.

As for the space between the vortices, we must have the equation of continuity satisfied; that is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

as well as the vortex intensity equal to zero

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0.$$

The four conditions (63) are reduced to two conditions. It is thus sufficient to have two of the quantities (63) equal to zero in order to have a neutral state of equilibrium of a single vortex.

It is easy to see that the conditions (63) mean that the velocity (u, v) must have a maximum or minimum at the point where the vortex A is disposed.

When the relations (1) are satisfied, a small displacement of the vortex A does not produce a change of its velocity. But let us consider the influence that the displacement of the vortex A can have on the other vortices of the system, say on the vortex B , for example. Let us imagine, first, that the vortex A undergoes a displacement δr along AB only (see Fig. 41). Let I be the strength of the vortex A . Before its displacement the vortex A was producing in B a velocity normal to AB , that is, to r and equal to $I/2\pi r$, which, combined with the velocities that the other vortices of the system produce in B , keep this vortex in relative equilibrium. After its displacement, the displaced vortex A will produce in B a velocity equal to $I/2\pi(r + \delta r)$. We thus see that we can consider the displacement of the vortex A as producing in B an additional velocity equal to

$$(64) \quad \frac{I}{2\pi(r + \delta r)} - \frac{I}{2\pi r} = -\frac{I}{2\pi r^2} \delta r$$

which is normal to r (see Fig. 41). In the same manner it will be easily seen that the displacement δn of A normal to r will produce in B an additional velocity directed along r and equal to (see Fig. 42)

$$(65) \quad -\frac{I}{2\pi r^2} \delta n$$

If we now let B displace itself in the direction of the additional velocity communicated by the displacement of A , we will at once see that such a displacement of B will produce in A the additional velocities $-\frac{I}{2\pi r^2} \kappa \delta r$ along r $-\frac{I}{2\pi r^2} \kappa \delta n$ along the normal to r where κ is a factor of proportionality, which will be exactly inverse to the displacements δr and δn when the two vortices considered are of inverse sense (see Fig. 43), and which will have the same senses as δr and δn when the two vortices considered have the same senses (see Fig. 44).

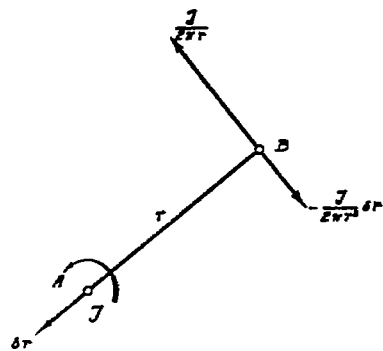


FIG. 41.

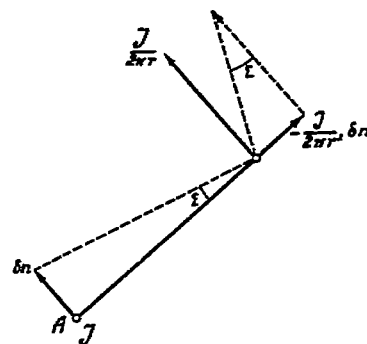


FIG. 42.

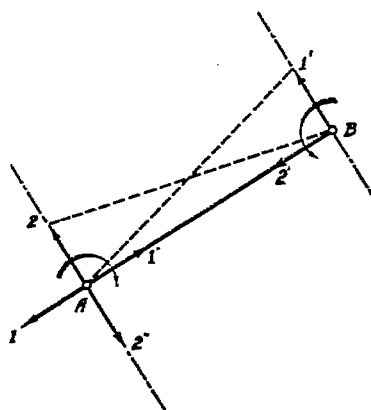


FIG. 43.

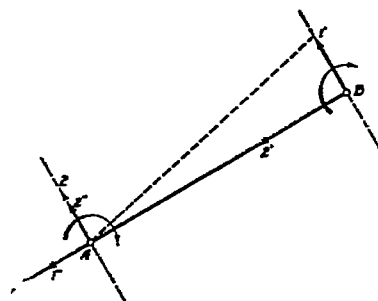


FIG. 44.

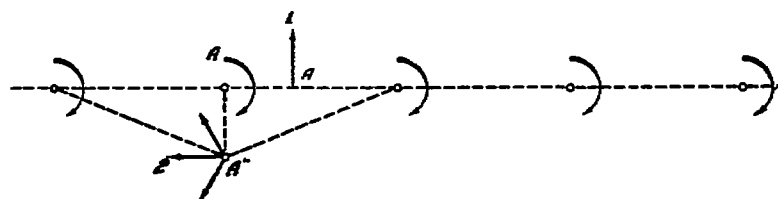


FIG. 45.

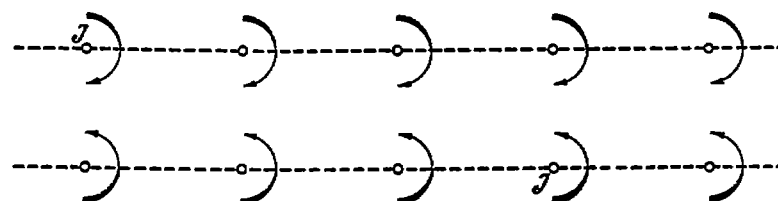


FIG. 46.

The foregoing preliminary discussion is given here only to show the nature of the question. It will, however, allow us to arrive at a general understanding of the stability conditions of vortex rows.

Let us consider first one vortex row, i. e., all the vortices in the same sense (see Fig. 45). If we displace one of the vortices of the row, say A , first to the position A' , considering the other vortices for one moment immobile, it will be easy to see that in the position A' this vortex will get from the other vortices an additional velocity in the sense of the arrow 1. If we displace this same vortex to the position A'' it will get an additional velocity in the sense of the arrow 2. Thus for neither displacement will there appear an additional velocity directed toward the original position of A . If we now consider the additional velocities which the vortex will receive from the displacements of the other vortices produced by the displacement of the vortex A , it will be easily seen from the foregoing, since all the vortices have the same senses, that these additional velocities will only increase the original displacement of the vortex A . Thus one vortex row appears as an unstable configuration.

Let us now consider two parallel vortex rows rotating oppositely in each row. The additional velocity given each displaced vortex will consist of the velocity due to its own displacement and of the velocities this vortex gets due to the displacements of the other vortices produced by its own displacement. This last additional velocity which is caused by the vortices of the same row as the vortex considered, has a destabilizing action, as all the vortices of the same row rotate in one sense, but the additional velocity caused by the vortices of the other row will produce a stabilizing action, as they rotate in inverse sense. We thus can conceive

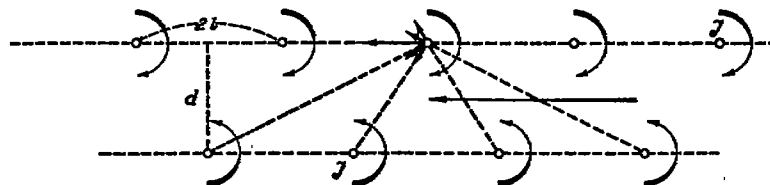


FIG. 47.

that two parallel rows of vortices with inverse rotation in each row can have a stable configuration. The investigations of Karman have shown that two parallel rows such as represented in Fig. 46 can not be stable, but two rows in quincunx as represented in Fig. 47 can be stable for a certain value of the ratio $d/2l = \lambda$ of the distance d between the two rows to the distance $2l$ between the vortices in each row. The most probable value of λ for a stable configuration of two vortex rows in quincunx seems to be¹

$$(66) \quad \frac{d}{2l} = \lambda = 0.283$$

If a single vortex row could be stable it is easy to see that it would be immobile, because the velocities which all the vortices communicate to one of them mutually balance. As for a quincunx arrangement of vortex rows, it is easy to see that it will move with a constant velocity parallel to the general direction of the rows (see Fig. 47) because each vortex of one row will receive no velocity from the vortices of the same row, but from each two vortices of the other row, disposed symmetrically relative to the vortex considered, it will receive a resultant velocity directed along the direction of the row.

Let us calculate the value of this velocity. For that purpose let us first calculate the velocity produced by a single vortex row in a point P . Let us refer the vortex row $\dots A', A_1, A_2, A_3, A_4, \dots$ to a system of orthogonal axes $X'O'Y'$ whose X' axis is parallel to the vortex row considered, and let us also consider the system of axes XOY parallel to the

¹ V. Karman has published two papers on that question. In his first paper "Über den Mechanismus des Widerstandes, den ein bewegter Körper in einer Flüssigkeit erfährt" Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 1911, p. 509, Karman found for λ the value 0.387. N. Joukowski has made some experiments which seem to verify this value. See his "Aérodynamique," Paris, 1916, p. 205. In his second paper "Über den Mechanismus des Flüssigkeits und Luftwiderstandes" Physikalische Zeitschrift, January 15, 1912, Karman finds for λ the value 0.283 and produces the results of experiments, also verifying this last value. This question thus demands further investigations.

system $X'O'Y'$ but whose origin O is midway between the vortices A'_1 and A (see fig. 48). Let (ξ, η) be the coordinates of the point P in reference to the axes $X'O'Y'$ and (x, y) the coordinates of the same point in reference to the XY axes. The coordinates of the origin O in reference to the axes $X'O'Y'$ will be designated by (ξ_0, η_0) . We have

$$(67) \quad x = \xi - \xi_0; \quad y = \eta - \eta_0$$

The coordinates of any vortex A of our row in reference to the $X'O'Y'$ axes will be designated by (a, b) . As is well known, we have¹

$$(68) \quad u - iv = \frac{I}{2\pi i} \sum \frac{1}{z - \alpha}$$

with

$$z = \xi + i\eta; \quad \alpha = a + ib; \quad i = \sqrt{-1}$$

and where u and v are the components of the velocity produced in the point P by the vortices of the row considered, I the intensity of each vortex of the row.

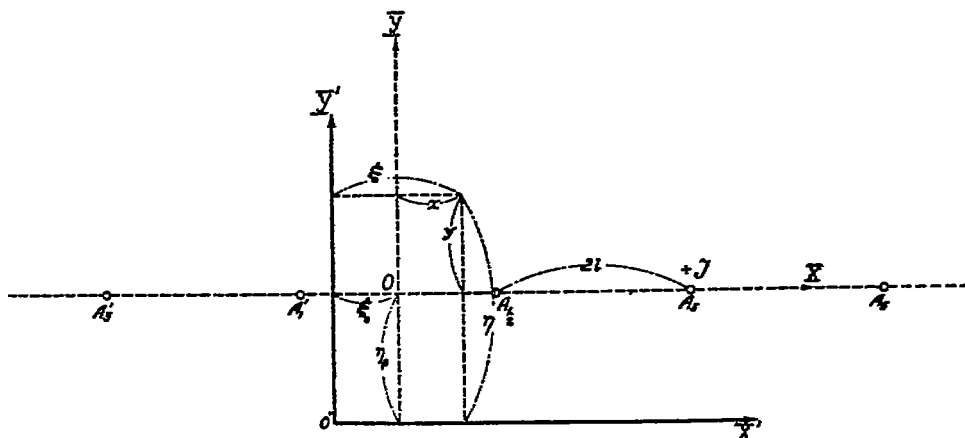


FIG. 48.

For the vortices of our row disposed symmetrically relative to O , the quantity α has for values:

$$\begin{array}{ll} \alpha_1 = (\xi_0 + l) + i\eta_0 & \alpha'_1 = (\xi_0 - l) + i\eta_0 \\ \alpha_2 = (\xi_0 + 3l) + i\eta_0 & \alpha'_2 = (\xi_0 - 3l) + i\eta_0 \\ \alpha_3 = (\xi_0 + 5l) + i\eta_0 & \alpha'_3 = (\xi_0 - 5l) + i\eta_0 \\ \text{-----} & \text{-----} \end{array}$$

Thus

$$\begin{array}{ll} z - \alpha_1 = (x - l) + iy & z - \alpha'_1 = (x + l) + iy \\ z - \alpha_2 = (x - 3l) + iy & z - \alpha'_2 = (x + 3l) + iy \\ z - \alpha_3 = (x - 5l) + iy & z - \alpha'_3 = (x + 5l) + iy \\ \text{-----} & \text{-----} \end{array}$$

and, consequently,

$$\frac{1}{z - \alpha_1} + \frac{1}{z - \alpha'_1} = \frac{2(x + iy)}{(x + iy)^2 - l^2} = \frac{2\rho}{\rho^2 - l^2}$$

with

$$\rho = x + iy$$

we thus see that

$$\sum \frac{1}{z - \alpha} = \sum_{k=1}^{k=\infty} \frac{2\rho}{\rho^2 - (2k-1)^2 l^2}$$

¹ See for example "Traité de Mécanique Rationnelle," by Paul Appell, Vol. III, p. 481.

If we now remember the well-known formula

$$\operatorname{tg} \theta = \sum_{k=1}^{k=\infty} \frac{4 \cdot 2\theta}{(2k-1)^2 \pi^2 - (2\theta)^2}$$

and identify this formula with

$$\sum_{k=1}^{k=\infty} \frac{2\rho}{\rho^2 - (2k-1)^2 l^2} = -\frac{\pi}{2l} \sum_{k=1}^{k=\infty} \frac{4 \cdot \frac{\pi\rho}{l}}{(2k-1)^2 \pi^2 - \frac{\rho^2 \pi^2}{l^2}}$$

it will be easily seen that with

$$\theta = \frac{\pi\rho}{2l}$$

we get

$$\sum_{k=1}^{k=\infty} \frac{2\rho}{\rho^2 - (2k-1)^2 l^2} = -\frac{\pi}{2l} \operatorname{tg} \frac{\pi\rho}{2l}$$

and thus

$$u - iv = -\frac{I}{4l} \operatorname{tg} \frac{\pi\rho}{2l}$$

or

$$\begin{aligned} -iu - v &= \frac{I}{4l} \operatorname{tg} \frac{\pi\rho}{2l} = \frac{I}{4l} \operatorname{tg} \frac{\pi(x+iy)}{2l} = \frac{I}{4l} \frac{\operatorname{tg} \frac{\pi}{2l} x + \operatorname{tg} \frac{\pi}{2l} iy}{1 - 2 \operatorname{tg} \frac{\pi}{2l} x \operatorname{tg} \frac{\pi}{2l} iy} \\ &= \frac{I}{4l} \frac{\operatorname{tg} \frac{\pi}{2l} x + i \operatorname{tgh} \frac{\pi}{2l} y}{1 - 2i \operatorname{tg} \frac{\pi}{2l} x \operatorname{tgh} \frac{\pi}{2l} y} = \frac{I}{4l} \frac{\operatorname{tg} x' + i \operatorname{tgh} y'}{1 - 2i \operatorname{tg} x' \operatorname{tgh} y'} \end{aligned}$$

introducing the notations

$$x' = \frac{\pi}{2l} x; y' = \frac{\pi}{2l} y$$

Thus

$$-iu - v = \frac{I}{4l} \frac{\operatorname{tg} x' + i \operatorname{tgh} y'}{1 - 2i \operatorname{tg} x' \operatorname{tgh} y'} = \frac{\operatorname{tg} x' (1 - 2 \operatorname{tgh}^2 y')}{1 + 4 \operatorname{tg}^2 x' \operatorname{tgh}^2 y'} + i \frac{\operatorname{tgh} y' (1 + 2 \operatorname{tg}^2 x')}{1 + 4 \operatorname{tg}^2 x' \operatorname{tgh}^2 y'}$$

and we finally find

$$(69) \quad v = \frac{I \operatorname{tg} x' (2 \operatorname{tgh}^2 y' - 1)}{4l (1 + 4 \operatorname{tg}^2 x' \operatorname{tgh}^2 y')} = \frac{I \operatorname{tg} \frac{\pi}{2l} (\xi - \xi_0) \left[2 \operatorname{tgh}^2 (\eta - \eta_0) \frac{\pi}{2l} - 1 \right]}{4l (1 + 4 \operatorname{tg}^2 \frac{\pi}{2l} (\xi - \xi_0) \operatorname{tgh}^2 \frac{\pi}{2l} (\eta - \eta_0))}$$

$$(70) \quad u = -\frac{I \operatorname{tgh} y' (1 + 2 \operatorname{tg}^2 x')}{4l (1 + 4 \operatorname{tg}^2 x' \operatorname{tgh}^2 y')} = -\frac{I \operatorname{tgh} \frac{\pi}{2l} (\eta - \eta_0) \left[1 + 2 \operatorname{tg}^2 \frac{\pi}{2l} (\xi - \xi_0) \right]}{4l (1 + 4 \operatorname{tg}^2 \frac{\pi}{2l} (\xi - \xi_0) \operatorname{tgh}^2 \frac{\pi}{2l} (\eta - \eta_0))}$$

These last formulae constitute the general expressions for the components u and v of the velocity at a point $P (\xi, \eta)$ produced by a single vortex row.

I will here note that, since for

$$(71) \quad \operatorname{tgh}^2 (\eta - \eta_0) \frac{\pi}{2l} = 1/2.$$

we have

$$v = 0$$

for any value of ξ , a single vortex row has around it such a flow that at the distance $(\eta - \eta_0)$ fixed by the relation (71), we have two streamlines that are straight lines,

If we now apply the formulae (69) and (70) to a system of two vortex rows in quincunx in order to find the velocity communicated to one row by the other, as we have $\xi = \xi_0$ for each vortex, we find

$$v = 0$$

and

$$(72) \quad u = -\frac{I}{4l} \operatorname{tgh} \frac{\pi}{2l} (\eta - \eta_0) = -\frac{I}{4l} \operatorname{tgh} \frac{\pi}{2l} d$$

where

$$\eta - \eta_0 = d$$

and for the stable configuration of two vortex rows in quincunx with $\lambda = d/2l = 0.283$ we find for the magnitude of u the value

$$(73) \quad u \cong 0.35 \frac{I}{2l}$$

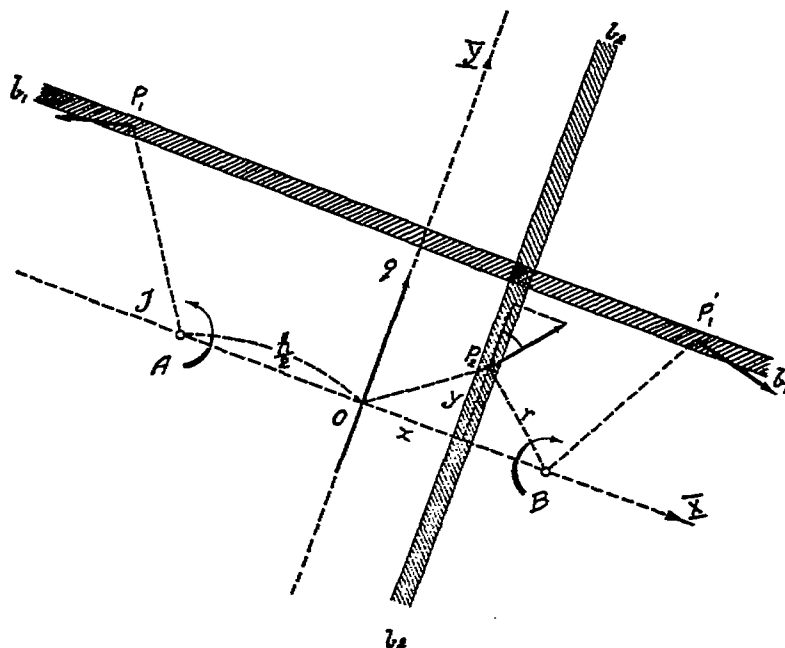


FIG. 49.

Let us further calculate the momentum q counted per unit of vortex length that corresponds to two vortices A and B of equal intensity I rotating in inverse senses which we will refer to the system of XOY axes. (See Fig. 49.) It is easy to see that the component of the momentum of the two vortices in the direction AB is equal to zero. Because if we consider a fluid strip parallel to the direction of AB such as b_1b_2 the component of the momentum along the direction of AB of the fluid element situated at a point such as p_1 is equal and directly opposite to the momentum of the fluid element p'_1 symmetrical to p_1 in reference to the axis Y (see Fig. 49). The component of the momentum along the direction of AB which corresponds to the whole strip b_1b_2 is thus equal to zero. As the same takes place for any fluid strip parallel to AB , the resultant momentum corresponding to the two vortices A and B will be normal to AB . For the calculation of this momentum, let us divide the fluid into strips normal to AB such as $b_2b'_2$. The component of the momentum of the fluid element situated at a point such as p_2 along the normal to AB , and due to one of the vortices is equal to (see Fig. 49)

$$\delta dx \, dy \cdot \frac{I}{2\pi r} \cdot \frac{h/2 - x}{r}$$

and for the whole strip considered

$$\delta dx \frac{I}{2\pi} \int_{-\infty}^{+\infty} \frac{(h/2-x)dy}{(h/2-x)^2+y^2} = \delta dx \frac{I}{2\pi} \int_{-\infty}^{+\infty} \frac{d\left(\frac{y}{h/2-x}\right)}{1+\frac{y^2}{(h/2-x)^2}}$$

when we note that

$$r^2 = (h/2-x)^2 + y^2$$

We thus find

$$\delta dx \frac{I}{2\pi} \int_{-\infty}^{+\infty} \frac{d\left(\frac{y}{h/2-x}\right)}{1+\frac{y^2}{(h/2-x)^2}} = \delta dx \frac{I}{2\pi} \left[\operatorname{arctg} \frac{y}{h/2-x} \right]_{-\infty}^{+\infty} = \frac{\delta I}{2} dx$$

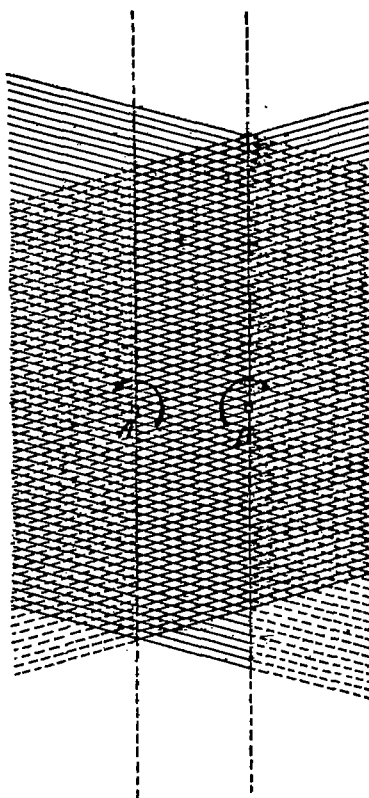


Fig. 50.

which quantity appears to be independent of the coordinate x . It will be easy to see that this last momentum is positive for the vortex B for all the strips at the left of B and negative for all the strips at the right of B . The inverse takes place for the vortex A (see Fig. 50). Thus when we calculate the total momentum q , the momentum of the strips outside A and B will mutually cancel, and there will be left the momentum of the strips between A and B . We thus find

$$(74) \quad q = 2 \frac{\delta I}{2} \cdot h = \delta I h$$

where h is the distance between the two vortices considered.

Let us now consider a system of two parallel rows of vortices in quincunx. We can always conceive this system as built up by the superposition of two identical vortex row systems with

the intensity of each vortex equal to $I/2$. (See Fig. 51.) To each pair of vortices of intensity $I/2$ will correspond a momentum normal to the line joining them and equal to

$$q = \delta h \frac{I}{2}$$

If we consider the sum of all this momentum for the whole system, it is easy to see that we will get only a resultant along the general direction of the rows. Each pair of vortices of intensity $I/2$ will contribute to this resultant by a momentum equal to

$$(75) \quad \delta h \frac{I}{2} \cdot \frac{d}{h} = \delta d \frac{I}{2}$$

and the resultant momentum counted per vortex of intensity I and per unit of length of the last will thus be equal to

$$\delta d \frac{I}{2}$$

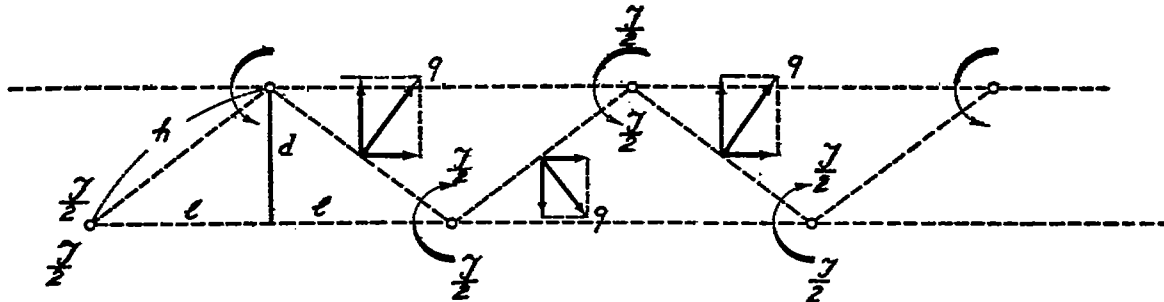


FIG. 51.

Summing up from the foregoing all the data relating to a system of two parallel vortex rows in quincunx we see that—

The quincunx system maintains a stable configuration for

$$(76) \quad \frac{d}{2l} = 0.283$$

The system communicates to itself a velocity parallel to the general direction of the rows equal in magnitude to

$$(77) \quad u = \frac{I}{4l} \operatorname{tgh} \frac{\pi d}{2l} \approx 0.35 \frac{I}{2l}$$

The resultant momentum of the system is directed along the general direction of the rows and counted per vortex and per unit of length of the last, is equal to

$$(78) \quad \delta d \frac{I}{2}$$

The Fundamental and Secondary Waves.—We will now make an attempt to calculate the order of magnitude of the fundamental and secondary waves. As we have seen in the foregoing, the fundamental wave will be produced by the vortices in quincunx built from the surfaces of discontinuity which are, in the limiting case, the boundaries of the wake behind a body or aerofoil. The secondary wave is produced by the vortices in quincunx built from the Kirchhoff-Lord Rayleigh surfaces of discontinuity. In some cases, in all probability only one of the waves will be formed; in others both waves will appear simultaneously and one will propagate itself in the other. The study of the conditions of formation of the fundamental and secondary waves demands further deep investigations. I will here consider only the cases in which each kind of wave appears separately.

Let us consider a solid body or an aerofoil disposed in a uniform fluid stream having a general velocity equal to V . If a system of two parallel vortex rows in quincunx is assumed to appear behind, the momentum counted per unit of time, corresponding to the vortices appearing, will be equal to the momentum per vortex, multiplied by the number of vortices formed. As the vortex in the flow will have a velocity equal to u the number of vortices formed will be equal to

$$2 \frac{V-u}{2l}$$

and the corresponding momentum equal to

$$(79) \quad \delta d I \frac{V-u}{2l}.$$

To a first approximation we can assume the drag of the body or aerofoil considered (counted per unit of length) equal to the last quantity—that is ¹

$$(80) \quad R_x = \delta d I \frac{V-u}{2l} = k_x \delta b V^2$$

or, introducing for u and d their values in (77) and (76), we get

$$(81) \quad k_x b = 0.283 \frac{I}{V} \left(1 - \frac{0.35}{2l} \frac{I}{V} \right)$$

The following considerations allow us to estimate the value that the intensity of the vortex in quincunx built behind a body or aerofoil may have.

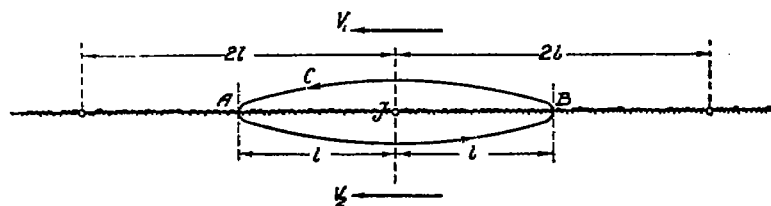


FIG. 52.

Let us consider in a fluid flow a surface of discontinuity, or vortex sheet, on both sides of which we have a finite velocity difference equal to $w' = V_1 - V_2$ (see fig. 52). For a contour such as C drawn between two points A and B of the surface, whose distance is $2l$, the circulation will have the value

$$2l V_1 - 2l V_2 = 2l (V_1 - V_2) = 2lw'$$

When now the vortex sheet considered goes over into a row of vortices whose mutual distance is $2l$, in the ideal case the circulation will remain invariable, and if a vortex is built between the points A and B , the intensity of it will be equal to

$$(82) \quad I = 2lw'$$

Thus in the case of the fundamental wave we will have

$$(83) \quad w' = w \text{ and } I = 2lw$$

where w is the mean wake velocity; and in the case of the secondary wave we will have

$$(84) \quad w' = V \text{ and } I = 2lV$$

the fluid inside the Kirchhoff-Lord Rayleigh surfaces of discontinuity being at rest and the velocity on the surface being equal to V .

¹ The whole purpose of this paper is to be first of all quite elementary. That is why I have allowed myself to give the foregoing formulae, which is not quite exact, the flow periodicity behind the body and the pressure difference in front and behind having been neglected. The error committed using this formula can reach 30 per cent; that is why this formula gives only an idea of the order of magnitude of the quantities considered, which is the only thing we wish to reach here.

Introducing the value (82) of I in (81) we find

$$k_x b = 0,283 \cdot 2l \frac{w'}{V} \left(1 - 0,35 \frac{w'}{V}\right)$$

and thus for the wave length $2l$ we get

$$(85) \quad 2l = \frac{k_x b}{0,283 \frac{w'}{V} \left(1 - 0,35 \frac{w'}{V}\right)}$$

Thus at first approximation the length of the fundamental wave appears equal to

$$(86) \quad 2l = \frac{k_x b}{0,283 \frac{w}{V} \left(1 - 0,35 \frac{w}{V}\right)}$$

and the length of the secondary wave equal to

$$(87) \quad 2l \cong 5,4 k_x b$$

To give a concrete example, let us take an aerofoil with

$$b = 2 \text{ mt.}; k_x = 0.05$$

and assume the ratio w/V to be of the order of 0,01.¹

In such a case, for the length of the fundamental wave we find

$$2l \cong \frac{0.01}{0.283 \cdot 0.01} \cong 35 \text{ mt.}$$

and thus

$$d = 0.283 \cdot 2l \cong 10 \text{ mt.}$$

For the length of the secondary wave we get

$$2l = 0.1 \cdot 5.4 \cong 0.5 \text{ mt}$$

and

$$d \cong 0.15 \text{ mt.}$$

Secondary waves have been observed by several experimentators, and are fully of the order of size given by the foregoing formulae. But fundamental waves, as far as the author has knowledge of the subject, have never been experimentally observed. The scale of this phenomenon shows the great interest of its experimental study. The possibility of the formation of the fundamental waves explains the action which bodies moving in a fluid may have on each other when they approach one another, as has been observed in some cases between airplanes and ships. The phenomenon of the fundamental wave indicates also how complicated is really the comparison of the fluid resistance of a body moving in a free and in a closed space. I will, finally, mention once more the fact that the phenomenon of the fundamental wave is a consequence of the fluid viscosity and can not be conceived in an ideal fluid.

The Pressure Distribution on the Surface of the Aerofoil.—One question has been left so far without discussion: It is the pressure distribution on the surface of the aerofoil. The general outlines of this phenomenon are easy to understand. The velocity of the flow running on the aerofoil is increased above the aerofoil and decreased below, which has as a direct consequence the decrease of pressure on the upper surface and the increase of pressure on the lower surface of the aerofoil. But we are not able to make the exact calculation of the pressure distribution along the surface of the aerofoil. The pressure distribution is very closely connected with the phenomena which take place on the surface of contact of the fluid and solid body. Until some new conception throws a new light on these phenomena, it does not seem that the pressure calculation can be started with any success. The general ideas which were developed in the preamble indicate the way in which the solution of the problem will probably be found.

¹ The value of w/V is connected with the value of k_x . The elementary considerations developed in the beginning of this chapter may be used to find the mutual order of magnitude of k_x and w/V .

NOTE I.—Generalization and General Discussion of Kutta's Theorem on Circulation.

The circulation theorem which I have in view in the present note was first indicated for a particular case by W. M. Kutta.¹ Soon afterwards, Kutta² and Joukowski³ have recognized the generality of this theorem. This theorem is announced as follows:

When a rectilinear and uniform fluid current, having at infinity the velocity \bar{V} , flows normally to the generatrix of an infinite cylinder from any section, and when the circulation along a contour embracing the cylinder and situated in the plane of one of its orthogonal sections has a finite value I , the component R_y of the resultant force of the fluid on the cylinder, taken along the normal to the velocity and referred to a unit length of the last, is equal to the product of the velocity V , the circulation I and the density of the fluid: The sense from R_y to \bar{V} is coincident with the sense of the circulation.

According to this theorem, the lift produced by a unit length of the cylinder considered is expressed in magnitude by the following formula:

$$R_y = \rho V I$$

We shall establish two fundamental and quite general relations from which the circulation theorem will appear as a particular case.

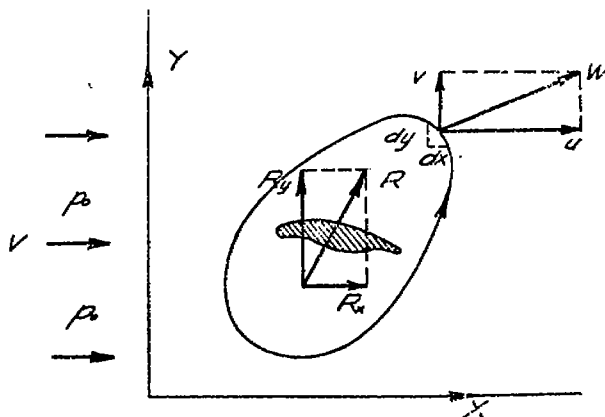


FIG. 1.

Let us embrace the infinite cylinder considered by any contour disposed in the plane of one of its orthogonal sections. Let W be the velocity of the fluid at the point (x, y) of the contour; u and v the components of the velocity W along the axes (see fig. 1); dx and dy the projections of the element of the contour on the axes. Let us designate by X and Y the components of the resultant force of all the exterior forces applied to the fluid contained in the contour considered and let us apply the theorem of momentum to the motion of the portion of the fluid considered. We then have

$$(1) \quad Y = \oint v dm; \quad X = \oint u dm,$$

the integral being taken around the contour and dm representing the fluid mass which flows out per unit of time through an element of the contour. Let us designate by ψ the current function. By the definition of that function, we have

$$(2) \quad dm = \delta d\psi$$

and also

$$(3) \quad d\psi = u dy - v dx = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx$$

with

$$(4) \quad u = \frac{\partial \psi}{\partial y}; \quad v = -\frac{\partial \psi}{\partial x}$$

Substituting in the first of the equations (1) the value of dm taken from the equation (2) we obtain

$$(5) \quad Y = \oint \delta v d\psi = \oint \delta v (u dy - v dx)$$

or,

$$Y = \oint \delta [v(u dy - v dx) + u^2 dx - v^2 dy] \\ = \oint \delta u (u dx + v dy) - \oint \delta (u^2 + v^2) dx;$$

and, remarking that

$$(6) \quad u dx + v dy = dI$$

¹ W. M. Kutta. *Illustrierte Aeronautische Mitteilungen*, 1902.

² W. M. Kutta. *Sitzungsberichte der Königlich Bayerischen Akademie der Wissenschaften*, München, 1910 and 1911.

³ N. E. Joukowski. *Geometrische Untersuchungen über die Kutta'sche Strömung*, Moscow, 1910, 1911. See also his course "Aérodynamique," Paris, 1916, p. 139.

is nothing else than the flow dI along an element of our contour, in a counterclockwise direction, i. e., a direction such as will turn the axis of X into that of Y , we get

$$(7) \quad Y = \int \delta u dI - \int \delta W^2 dx$$

and finally, integrating by parts the first term of the second member of that relation, we get

$$(8) \quad Y = \left[\delta u I - \int \delta I du \right] - \int \delta W^2 dx$$

which expression holds for any contour and constitutes the first of the relations we wished to get

Applying that relation (8) to a contour along which

$$v=0; u=V=\text{Const.}$$

we easily see that we have

$$(9) \quad \int \delta I du = 0; \int \delta W^2 dx = 0$$

and consequently Y reduces to

$$(10) \quad Y = \delta VI$$

I being the circulation along the contour considered.

Following the same way with the second of the equations (1), we get

$$(11) \quad \begin{aligned} X &= \int \delta u dy = \int \delta u (udy - vdx) \\ &= \int \delta (u^2 dy - uv dx - v^2 dy + v^2 dy) \\ &= \int \delta (u^2 + v^2) dy - \int \delta v (u dx + v dy) \end{aligned}$$

$$(12) \quad X = \int \delta W^2 dy - \int \delta v dI$$

$$(13) \quad X = \int \delta W^2 dy - \left[\delta v I - \int \delta I dv \right]$$

The last of these equations constitutes the second relation we wished.

Applying this last relation to a contour along which

$$v=0; u=V=\text{Const.}$$

we easily see that we have

$$(14) \quad X = 0;$$

all the three terms of the second member of the relation (13) being equal to zero.

Let us now stop to note the exact interpretation of the relations (10) and (14). As it has been indicated, X and Y are the components of the resultant forces of all the exterior forces applied to the volume of fluid contained in the contour considered. These forces are: First, the pressures of the cylinder on the fluid, which are equal and opposite to the pressures of the fluid on the cylinder; second, the exterior pressures on the contour. Let us consider a contour over which $v=0$; $u=V=\text{Const.}$ and which is limited in one sense by two stream lines sufficiently distant from the cylinder so that they are parallel to the X axis, and in the other sense by two lines perpendicular to these stream lines. Along the stream lines parallel to the X axis we can consider the Bernoulli constant as being effectively constant, and in consequence the pressure p constant and equal to the exterior pressure p_0 , the velocity V being constant. Under this condition the component along the Y axis from the exterior pressures on our contour will be zero, and Y will represent the negative of the component of the pressures of the fluid on the immersed cylinder. The expression (10) consequently gives us for the numerical value R_y of the lift of the fluid on our cylinder $R_y = \delta VI$. But if we consider a stream line which flows near our cylinder, there must be some interior loss through viscosity along this stream line because each immersed body gives rise to drag. The Bernoulli constant along such a stream line must necessarily decrease, and when we reach the side of the contour, parallel to the Y axis, where the velocity V has again taken its original value, the pressure there will not take its original value p_0 , the Bernoulli constant having decreased. The relation (14) consequently expresses the fact that the component along the X axis of the resultant force of the exterior pressures on our contour is exactly equal to the drag, and this still in the case when the sides of our contour are moved to infinity. In the last case, the exterior pressures tend to their limit value p_0 , but this is not reached; and the integral

$$\int p dy$$

always remains exactly equal to the drag. Messrs. Kutta and Joukowski, who were the first to establish the relations (10) and (14), have limited themselves to a consideration of a perfect fluid. In that case, having no interior losses, the Bernoulli constant has an invariable value along any stream line and the relation (14) expresses, then, the fact that the drag of an immersed cylinder is zero. But it is absolutely unnecessary to limit ourselves to a perfect fluid, since the theorem of momentum from which the equations (10) and (14) are a direct consequence, is applicable whatever the interior forces acting on the considered system are.

We are thus brought to the general conclusion that, for any contour surrounding an immersed cylinder, the following general relations must hold:

$$(15) \quad \int p dx - R_y = \int \delta v (udy - vdx) = \int \delta u dI - \int \delta W^2 dx = \left[\delta u I - \int \delta I du \right] - \int \delta W^2 dx$$

$$(16) \quad \int p dy - R_x = \int \delta u (udy - vdx) = \int \delta W^2 dy - \int \delta v dI = \int \delta W^2 dy - \left[\delta v I - \int \delta I dv \right]$$

which connect the lift and drag of the cylinder, referred to a unit length of the last, with the flow around this cylinder.

In the application of these formulas, three particular cases have to be distinguished:

1st. The formulas are applied to the contour of the cylinder itself. The contour of the cylinder being a stream line through which we have no flow, we must simply have

$$R_y = \int p dx; \quad R_x = \int p dy$$

which is the case considered in classical hydrodynamics.

2d. The formulas are applied to a contour which consists of stream lines and normal lines. (For the definition of the last lines, see Note II.) In that case the integrals which figure in the second members of the relations (15) and (16) have to be calculated only along the normal lines.

3d. The Kutta case

$$R_y = -\delta VT; \quad R_x = \int p dy$$

NOTE II.—Generalization of the Bernouilli Theorem.

For the determination of the pressures in a fluid, we have the Bernouilli theorem, which furnishes us the law of variation of pressure along a stream line and also along a vortex line. We also know that the Bernouilli theorem is applicable to the whole fluid, considering the Bernouilli constant as invariable when the fluid motion is irrotational. But in the general case, when we go from one stream line to another, the Bernouilli constant changes its value. What is the law of variation of the Bernouilli constant in the whole fluid mass in the general case? It is the solution of this important question which the present note gives. We so arrive at the general solution of the problem of the distribution of pressures in a fluid mass.

Let us consider a fluid mass in a steady state of motion. Let us consider in this fluid mass the *stream line curves* and also two other families of fundamental curves: the *normal lines*, defined by the property that the tangent at each point to those curves coincides with the principal normal of the stream line passing through this point, and the *binormal lines*, defined by the property that the tangent at each point coincides with the binormal to the corresponding stream line. The stream lines, the normal lines, and the binormal lines form a system of triorthogonal curves.

These curves have for equations:

The stream lines:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

The normal lines:

$$\frac{dx}{Bw - Cv} = \frac{dy}{Cu - Aw} = \frac{dz}{Av - Bu}$$

The binormal lines:

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}$$

In these equations u, v, w are the components of the velocity of the fluid and A, B, C the determinants of the matrix:

$$\begin{vmatrix} u & v & w \\ \frac{du}{dt} & \frac{dv}{dt} & \frac{dw}{dt} \end{vmatrix}$$

For example

$$A = v \frac{dw}{dt} - w \frac{dv}{dt}$$

expressions in which $\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$ are the total derivatives; for example,

$$\frac{du}{dt} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

the motion being steady.

Let us consider a fluid element contained in the elementary parallelepiped, whose edges $d\tau, d\nu, d\beta$ are respectively directed along the stream lines, the normal lines and the binormal lines. On these curves, we get the following positive senses: On the stream lines, the sense of the velocity of the fluid particles; on the normal lines, the sense toward the center of curvature of the corresponding stream lines; on the binormal lines, the positive sense is chosen in such a way that the trirectangular trihedral ($d\tau, d\nu, d\beta$) be positive.

Let us apply the d'Alembert principle to the fluid element $d\tau, d\nu, d\beta$ and let us consider for the sake of simplicity the fluid as incompressible and having no weight (see fig. 1). The resultant of the exterior pressure on the fluid element has for components:

$$\begin{aligned} & -\frac{\partial p}{\partial \tau} d\tau d\nu d\beta \text{ along } d\tau \\ & -\frac{\partial p}{\partial \nu} d\tau d\nu d\beta \text{ —. — } d\nu \\ & -\frac{\partial p}{\partial \beta} d\tau d\nu d\beta \text{ —. — } d\beta \end{aligned}$$

p being the pressure at the point considered.

The resultant of the forces of inertia applied to that element has for components:

$$\begin{aligned} & -\frac{dV}{dt} \delta d\tau d\nu d\beta \text{ along } d\tau \\ & -\frac{V^2}{\rho} \delta d\tau d\nu d\beta \text{ —. — } d\nu \\ & \quad \quad \quad 0 \text{ —. — } d\beta \end{aligned}$$

δ being the density of the fluid at the point considered; V the velocity, and ρ the radius of the principal curvature.

According to the d'Alembert principle, we must have:

$$\begin{aligned} (1) \quad & \frac{\partial p}{\partial \tau} + \delta \frac{dV}{dt} = 0 \\ (2) \quad & \frac{\partial p}{\partial \nu} + \delta \frac{V^2}{\rho} = 0 \\ (3) \quad & \frac{\partial p}{\partial \beta} = 0 \end{aligned}$$

This system of relations represents the equation of motion of the fluid referred to the triorthogonal curvilinear system of stream lines, normal lines and binormal lines, which can be called the *natural curvilinear coordinates of the fluid* or, shorter, the *natural coordinates of the fluid*.

I. The equation (1) brings us directly to the Bernoulli theorem. We have

$$\frac{\partial p}{\partial \tau} + \delta \frac{dV}{dt} = \frac{\partial p}{\partial \tau} + \delta \frac{dV}{d\tau} \frac{\partial \tau}{\partial t} = \frac{\partial p}{\partial \tau} + \delta V \frac{dV}{d\tau} = 0$$

and integrating along a stream line, we get

$$(4) \quad p + \frac{\delta V^2}{2} = C$$

a relation which constitutes the Bernoulli theorem, C being the Bernoulli constant.

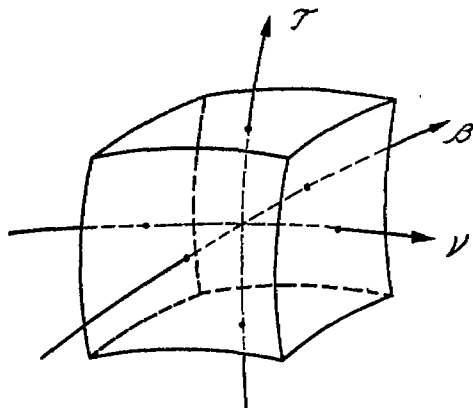


FIG. 1.

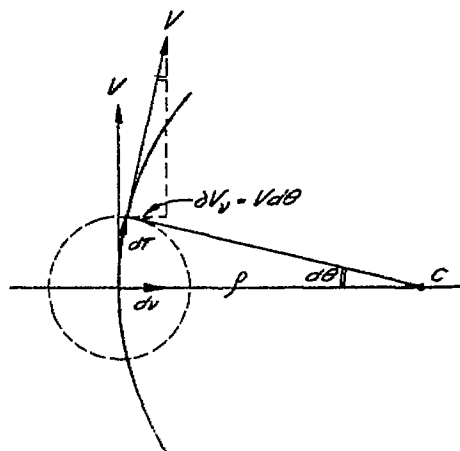


FIG. 2.

II. The equation (2) gives us the distribution of pressure along the normal lines. Integrating this equation along the normal lines, we get

$$(5) \quad p + \int \delta \frac{V^2}{\rho} d\nu = C$$

This last equation is susceptible of the following important transformation:

Let us designate by $\omega_r, \omega_\nu, \omega_\beta$ the components of the vortex and by V_r, V_ν, V_β the components of the resultant velocity V along the directions $d\tau, d\nu, d\beta$ at the point considered. We have

$$V_r = V; V_\nu = 0; V_\beta = 0$$

The relations between the double of the component of the vortex ω and the component of the velocity are given by the determinants of the matrix

$$(6) \quad \begin{vmatrix} \frac{\partial}{\partial \tau} & \frac{\partial}{\partial \nu} & \frac{\partial}{\partial \beta} \\ V_r & V_\nu & V_\beta \end{vmatrix}$$

We thus have

$$(7) \quad 2\omega_\beta = \frac{\partial V_r}{\partial \tau} - \frac{\partial V_\tau}{\partial \nu}$$

or

$$(8) \quad \frac{\partial V_r}{\partial \tau} - 2\omega_\beta = \frac{\partial V_r}{\partial \nu} = \frac{\partial V}{\partial \nu}$$

On the other hand (see fig. 2).

$$(9) \quad \frac{\partial V_r}{\partial \tau} = \frac{V d\theta}{d\tau} = \frac{V}{\rho}$$

$d\theta$ being the contingence angle. Substituting this last value of $\frac{\partial V_r}{\partial \tau}$ in the relation (8), for an integration along a normal line, we get

$$(10) \quad d\nu = \frac{dV}{\frac{V}{\rho} - 2\omega_\beta}$$

and substituting this last value of $d\psi$ in the equation (5), we get:

$$(11) \quad p + \int \frac{\delta V dV}{1 - 2\rho \frac{\omega \beta}{V}} = C$$

Finally, the integral of this last relation is susceptible of the following transformation

$$\begin{aligned} \int \frac{\delta V dV}{1 - 2\rho \frac{\omega \beta}{V}} &= \int \frac{\delta V dV \left[1 - \left(1 - 2\rho \frac{\omega \beta}{V} \right) + \left(1 - 2\rho \frac{\omega \beta}{V} \right) \right]}{1 - 2\rho \frac{\omega \beta}{V}} \\ &= \int \delta V dV - \int \frac{\delta V dV \cdot 2\rho \frac{\omega \beta}{V}}{1 - 2\rho \frac{\omega \beta}{V}} \\ &= \frac{\delta V^2}{2} - \int \frac{\delta V dV}{\frac{2\rho \omega \beta}{V} - 1} \end{aligned}$$

and the equation (11) takes the form

$$(12) \quad p + \frac{\delta V^2}{2} = C + \int \frac{\delta V dV}{\frac{2\rho \omega \beta}{V} - 1}$$

which relation gives the distribution of pressure along the normal lines.

We easily see that the last equation has the form of the Bernouilli equation, only the integral which figures in the second member determines the variation of the Bernouilli constant when we go from one stream line to another along a normal line.

If we put

$$(13) \quad \Delta C = \int \frac{\delta V dV}{\frac{2\rho \omega \beta}{V} - 1}$$

the equation (12) takes the form

$$(14) \quad p + \frac{\delta V^2}{2} = C + \Delta C$$

We now see, it is sufficient for $\omega \beta$ to equal zero along a normal line—which means that on the normal line considered the vortex ω is in the contingency plane—for the integral ΔC to be equal to zero and therefore for the Bernouilli constant to be invariable along the normal line considered. It is evident that $\omega \beta$ is zero when $\omega = 0$.

The integral ΔC can be written in the form

$$(15) \quad \Delta C = \int \frac{\delta dV}{\frac{1}{2\rho \omega \beta} - \frac{1}{V}}$$

and is then susceptible of the following geometrical interpretation: The denominator of this integral represents the difference between the inverse of the speed which the fluid particle would have if rotating with the angular velocity $2\omega \beta$ around the center of curvature of its instantaneous position and the inverse of the resulting velocity V of the particle.

III. The integration of the equation (3) along the binormal lines brings us directly to the conclusion that along those lines

$$(16) \quad p = \text{Const.}$$

that is to say, in the case of a nonheavy fluid, the binormal lines are isobars. It will be easily seen that in the case of a heavy fluid, the distribution of pressure along the binormal lines will be the same as if the fluid were immobile.

We also see that for the case of irrotational motion of a fluid the binormal lines are also the lines of constant velocity, the Bernouilli theorem being applicable to the whole fluid mass.

The system of relations for (11), (12), and (16) fully determines in the general case the distribution of pressures in a fluid mass in motion. This system of relations carries us to the following important consequences, which I will indicate in general outlines:

I. It is sufficient to know the distribution of pressure along a surface cutting all the binormal lines in order to know the distribution of pressure in the whole fluid mass.

This proposition is a direct consequence of the fact that the pressure is constant along a binormal line.

II. On both sides of a vortex layer, even thin, there can exist a difference of pressure which can be of sensible value.

To convince ourselves of such a possibility, it is enough to picture a vortex layer in which the quantity

$$V - 2\rho \omega \beta$$

has a small value inside the layer, which can happen without V and $\omega\delta$ having excessive values. Then, when traversing the layer along a normal line, the integral

$$\int \frac{\delta V^2 dN}{V - 2\rho\omega\delta}$$

will have a large value, and, consequently, according to formula (11), the difference of pressure on the two sides of the layer can have a sensible value.

The conception of a thin vortex layer maintaining a sensible difference of pressure is very important for the understanding of many hydrodynamical phenomena which take place in real fluids. I shall take a typical example.

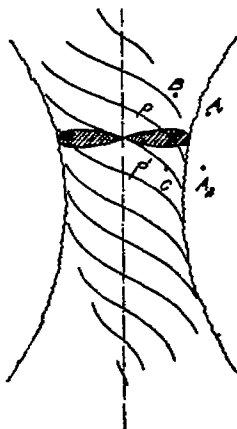


FIG. 3.

Let us consider a propeller having any number of blades and working in free air at a fixed point, for example. The rotation of the propeller creates a well-enough limited fluid stream. Let us follow a stream line in the sense of the motion of the fluid projected by the propeller (see fig. 3). When we reach a point such as B disposed before the propeller, the pressure p must be necessarily less than the exterior pressure p_0 , because the velocity is all the time increasing when we approach the propeller and at points such as A_1 and A_2 we have pressures very close to the pressure p . But when we go through the plan of rotation of the screw, the pressure increases and in a point such as C disposed directly behind the propeller, the pressure p' is generally greater than the exterior pressure p_0 . It would be difficult to conceive the existence of different pressures p' and p at points C and A_2 , if it were not for the vortex layer, which consequently must constitute the surface of the fluid stream created by the screw and which is capable of maintaining differences of pressure. Without the knowledge of the existence of the vortex layer constituting the surface of the stream created by the propeller the distribution of pressure around the propeller would be difficult to conceive.

NOTE III.—The Equation of the Metacentric Curve.

The aerofoil considered is referred to any system of X and Y axes invariably connected with the aerofoil, for example, the one represented on Fig. 1. The air resistance R of the aerofoil is resolved into two components R^1_x and R^1_y along the X and Y axes. These components are connected with the drag R_x and lift R_y of the aerofoil by the relations

$$(1) \quad R^1_x = R_x \cos \alpha - R_y \sin \alpha$$

$$(2) \quad R^1_y = R_x \sin \alpha + R_y \cos \alpha$$

Let us designate by N the moment of R relative to the origin O . We have

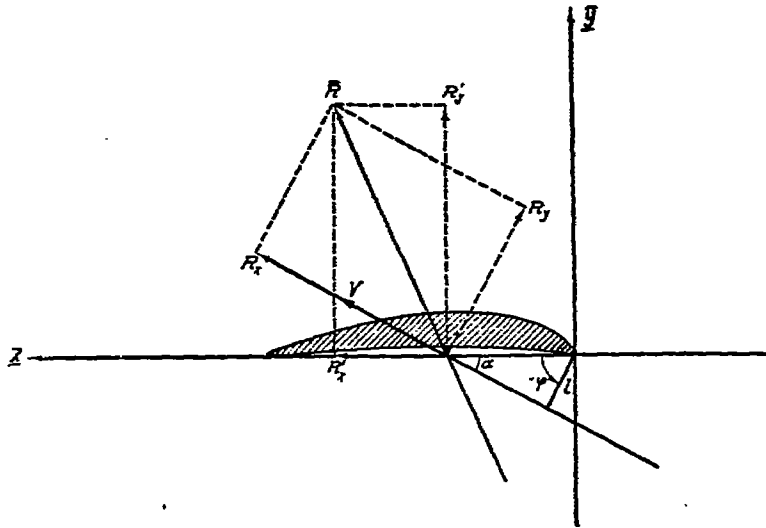
$$N = R \cdot l.$$

(3) The quantities

$$R^1_x; R^1_y; N$$

fully define the vector R in position and direction and are sometimes called the Plucker's coordinates of a vector. In these coordinates the equation of the direction of R is

$$(4) \quad x R^1_y - y R^1_x - N = 0$$



In fact, the normal form of the equation of the direction of R is

$$x \cos \varphi + y \sin \varphi - l = 0$$

but

$$R^1_y = R \cos \varphi; R^1_x = -R \sin \varphi$$

and thus

$$\cos \varphi = \frac{R^1_y}{R}; \sin \varphi = -\frac{R^1_x}{R}$$

consequently

$$x \frac{R^1_y}{R} - y \frac{R^1_x}{R} - l = 0$$

or

$$x R^1_y - y R^1_x - Rl = 0$$

and the equation (4) is thus established.

The metacentric curve is the envelope of the consecutive positions of the air resistance \bar{R} , and thus is fixed by the system of relations

$$(5) \quad \begin{cases} x R^1_y - y R^1_x - N = 0 \\ x \frac{R^1_y}{\partial \alpha} - y \frac{\partial R^1_x}{\partial \alpha} - \frac{\partial N}{\partial \alpha} = 0 \end{cases}$$

(6)

from which we find

$$(8) \ x = \frac{\begin{vmatrix} N & -R^1_x \\ \frac{\partial N}{\partial \alpha} & \frac{\partial R^1_x}{\partial \alpha} \end{vmatrix}}{\begin{vmatrix} R^1_y & -R^1_x \\ \frac{\partial R^1_y}{\partial \alpha} & \frac{\partial R^1_x}{\partial \alpha} \end{vmatrix}} \quad (9) \ y = \frac{\begin{vmatrix} R^1_y & N \\ \frac{\partial R^1_y}{\partial \alpha} & \frac{\partial N}{\partial \alpha} \end{vmatrix}}{\begin{vmatrix} R^1_y & -R^1_x \\ \frac{\partial R^1_y}{\partial \alpha} & \frac{\partial R^1_x}{\partial \alpha} \end{vmatrix}}$$

which represent the equation of the metacentric curve in parametric form, α being the parameter.

These equations can also be written

$$x = \frac{R^1_x \frac{\partial N}{\partial \alpha} - N \frac{\partial R^1_x}{\partial \alpha}}{R^1_x \frac{\partial R^1_y}{\partial \alpha} - R^1_y \frac{\partial R^1_x}{\partial \alpha}} - \frac{\frac{\partial}{\partial \alpha} \left(\frac{N}{R^1_x} \right)}{\frac{\partial}{\partial \alpha} \left(\frac{R^1_y}{R^1_x} \right)} - \frac{\frac{\partial}{\partial \alpha} \left[l \sqrt{1 + \frac{R^1_y{}^2}{R^1_x{}^2}} \right]}{\frac{\partial}{\partial \alpha} \left(\frac{R^1_y}{R^1_x} \right)}$$

$$y = \frac{R^1_y \frac{\partial N}{\partial \alpha} - N \frac{\partial R^1_y}{\partial \alpha}}{R^1_x \frac{\partial R^1_y}{\partial \alpha} - R^1_y \frac{\partial R^1_x}{\partial \alpha}} - \frac{\frac{\partial}{\partial \alpha} \left(\frac{N}{R^1_y} \right)}{\frac{\partial}{\partial \alpha} \left(\frac{R^1_x}{R^1_y} \right)} - \frac{\frac{\partial}{\partial \alpha} \left[l \sqrt{1 + \frac{R^1_x{}^2}{R^1_y{}^2}} \right]}{\frac{\partial}{\partial \alpha} \left(\frac{R^1_x}{R^1_y} \right)}$$

which can be used for the tracing of metacentric curves.

NOTE IV.¹—The Mechanism of Fluid Resistance.²

By TH. V. KÁRMÁN and H. RUBACH.

The resistance of a solid body moving with a uniform velocity in an unlimited fluid can be calculated theoretically only in the limiting cases of very slow motion of small bodies or of very high fluid viscosity. We are brought in such cases to a resistance proportional to the first power of the velocity, to the viscosity constant, and, for geometrically similar systems, to the linear dimensions of the body. To the domain of this "linear resistance"—which has aroused much interest, especially within recent years, on account of some important experimental applications—has to be opposed the limiting domain of comparatively large velocities, for which the so-called "velocity square law" holds with very good approximation. In this latter domain, which embraces nearly all the important technical applications, the resistance is nearly independent of fluid viscosity, and is proportional to the fluid density, the square of the velocity, and—again for geometrically similar systems—to a surface dimension of the body. In this domain of the "square law" is included the important case of air resistance, because it is easy to verify, by the calculation of the largest density variations which can occur for the speeds we meet in aeronautics and airscrews, that the air compression can be neglected without any sensible error. The influence of the compression first becomes important for velocities of the order of the velocity of sound. In fact, experiments show that the air resistance, in a broad range from the small speeds at which the viscosity plays a role up to the large speeds comparable to the velocity of sound, is proportional to the square of the velocity with very good approximation.³ In general, fluid resistance depends upon the form and the orientation of the body in such a complicated way that it is extraordinarily difficult to predetermine the flow to a degree sufficient for the evaluation of the resistance of a body of given form, by a process of pure calculation, as can be done by aid of the Stokes formula in the case of very slow motions. We also will not succeed in this paper in reaching such a solution, but will still make the attempt to give a general view of the *mechanism of fluid resistance within the limits of the square law*.

We can state the problem of fluid resistance in the following somewhat more exact way.

Since the time of the fundamental considerations of *Osborne Reynolds* on the mechanical similitude of flow phenomena of incompressible viscous fluids of different density and viscosity and—under geometrical similitude—for different sizes of the system considered, it is known that the resistance phenomenon depends upon a single parameter which is a certain ratio of the above-mentioned quantities. Thus the fluid resistance of a body moving with the uniform velocity U in an incompressible unlimited fluid may be expressed by a formula of the form ⁴

$$(I) \quad W = \mu l U f\left(\frac{Ul\rho}{\mu}\right)$$

where

μ is the viscosity constant

ρ the fluid density

l a definite but arbitrarily chosen linear dimension of the body, and $f\left(\frac{Ul\rho}{\mu}\right)$ a function of the single variable

$R = \frac{Ul\rho}{\mu}$. We will call "Reynolds' parameter" the quantity R which has a zero dimension.

Theory and experiment show that for very small values of R —that is, for low velocities, or small bodies, or great viscosity—the function $f(R)$ is very nearly constant; the resistance coefficient of the Stokes formula corresponds to the limiting case of $f(R)$ for $R=0$. The square law corresponds to the limiting case of $R=\infty$. We approach this latter case the more nearly the smaller the viscosity μ , so that in the limiting case of $R=\infty$, the fluid can be considered as frictionless. And we can ask ourselves, *to what limiting configuration does the flow of the viscous fluid around a solid body tend when we pass to the limiting case of a perfect fluid?* This is, according to our view, the fundamental point of the resistance problem.

The fact that we obtain in this case a resistance nearly independent of the viscosity constant—since according to formula (I) this corresponds to the square law—allows us to conjecture that in this limiting case the resistance is determined by flow types such as can occur in a perfect fluid.

¹ This note is the translation of the paper of Th. v. Kármán and H. Rubach published in "Physikalische Zeitschrift," Jan. 15, 1912. The author has considered it necessary to add here this complete translation, on account of the importance of the new conceptions of Th. v. Kármán.

² The actual note constitutes a more complete exposition of two notes of Th. v. Kármán published in the "Nachrichten der Kgl. Gesellsch. der Wiss. zu Göttingen," containing his hydrodynamical researches upon the stability of vortex systems and the conclusions concerning fluid resistance obtained from the latter. The experiments here discussed and the measurements given have a provisional character; exact measurements are expected in connection with an intended dissertation of H. Rubach.

³ See, for example, the interesting experiments of O. Föppl on the validity of the square law for air resistance, "Dissertation," Aachen, p. 40 (also "Jahrbuch der Motorluftschiff-Studiengesellschaft," 1911).

⁴ Compare Lord Rayleigh, Phil. Mag., vol. 21, p. 708, 1911.

It is now certain that neither the so-called "continuous" potential flow, nor the "discontinuous" potential flow discovered by Kirchhoff and v. Helmholtz, can express properly this limiting case. Continuous potential flow does not cause any resistance in the case of uniform motion of a body, as may be shown directly by aid of the general momentum theorem; the theory of the discontinuous potential flow, which, in relation to the resistance problem has been discussed principally by Lord Rayleigh,¹ leads to a resistance which is proportional to the square of the velocity; the calculated values do not, however, agree with the observed ones. And, independent of the insufficient agreement between the numerical values, the hypothesis of the "dead water," which, according to this theory ought to move with the body, is in contradiction to nearly all observations. It is easy to see by aid of the simplest experiments that the flow, when referred to a system of coordinates moving with the body, is not stationary, as assumed in this theory. Furthermore, in the theory of discontinuous potential motion, the suction effect behind the body is totally missing, while in the dead water, which extends to infinity, we have everywhere the same pressure as in the undisturbed fluid at a great distance from the body. But according to recent measurements, in many cases the suction effect is of first importance for the resistance, and in any case contributes a sensible part of the last.

The reason why in a perfect fluid the discontinuous potential flow, although hydrodynamically possible, is not realized is without any doubt the instability of the surfaces of discontinuity, as has already been recognized by v. Helmholtz and specially mentioned by Lord Kelvin.² A surface of discontinuity can be considered as a vortex sheet; and it can be shown in a quite general way that such a sheet is always unstable. This can also be observed directly; observation shows that vortex sheets have a tendency to roll themselves up; that is, we see the concentration around some points of the vortex intensity of the sheet originally between them. This observation leads to the question: Can there exist

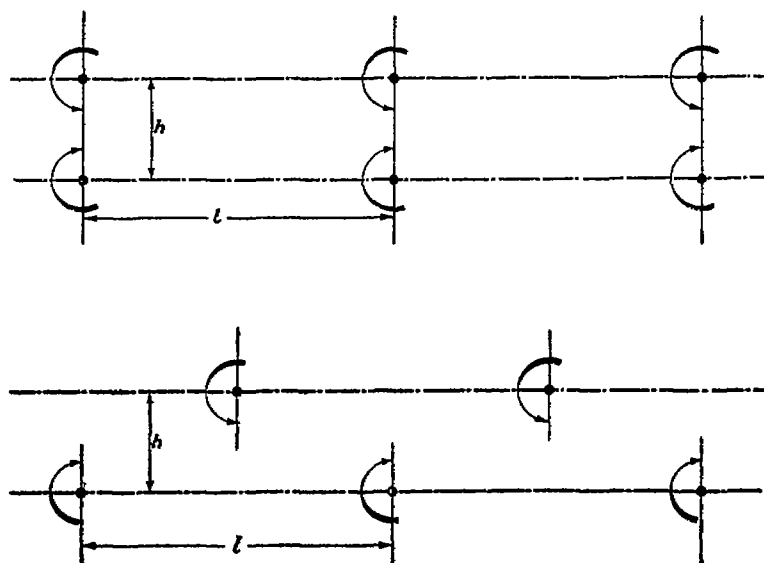


FIG. 1.

stable arrangements of isolated vortex filaments, which can be considered as the final product of decomposed vortex sheets? This question forms the starting point of the following investigations; it will, in fact, appear that at least for the simplest case of uniplanar flow, to which we will limit ourselves, we will be led to a "flow picture" which in all respects corresponds quite well to reality.

THE INVESTIGATION OF STABILITY.

We will investigate the question whether or not two parallel rows of rectilinear infinite vortices, of equal strength but of inverse senses, can be so arranged that the whole system, while maintaining an invariable configuration, will have a uniform translation and be stable at the same time. It is easy to see that there exist two kinds of arrangements for which two parallel vortex rows can move with a uniform and rectilinear velocity. The vortices may be placed one opposite the other (arrangement a, fig. 1), or the vortices of one row may be placed opposite the middle points of the spacing of the vortices of the other row (arrangement b). In the case of equality of spacing of the vortices in both rows, as a consequence of symmetry for the two arrangements a and b, it appears that each vortex has the same velocity in the sense of the X axis, and that the velocity in the sense of the Y axis is equal to zero. We have to answer the question, which of these two arrangements is stable?

To illustrate first by a simple example the method of the investigation of stability, we will start with the consideration of an infinite row of infinite vortices disposed at equal distances l and having the intensity Γ , and will study

¹ On the resistance of fluids, Mathematical and Physical Papers, Vol. I, p. 287.

² Mathematical and Physical Papers, Vol. IV, p. 215. This paper contains a detailed critique of the theory of discontinuous motion.

the stability of such a system. If we designate by x_p, y_p , the coordinates of the p -th vortex, and by x_q, y_q the coordinates of the q -th the velocity impressed on the latter vortex by the former is given by the formulæ

$$u_{pq} = \frac{\Gamma}{2\pi} \cdot \frac{y_p - y_q}{(x_p - x_q)^2 + (y_p - y_q)^2}$$

$$v_{pq} = -\frac{\Gamma}{2\pi} \cdot \frac{x_p - x_q}{(x_p - x_q)^2 + (y_p - y_q)^2}$$

These formulæ express the fact that each vortex communicates to the other a velocity which is normal to the line joining them and is inversely proportional to their distance apart. Therefore the resultant velocity of the q -th vortex due to all the vortices is equal to

$$\frac{dx_q}{dt} = \frac{\Gamma}{2\pi} \sum_{p=-\infty}^{\infty} \frac{y_p - y_q}{(x_p - x_q)^2 + (y_p - y_q)^2}$$

$$\frac{dy_q}{dt} = -\frac{\Gamma}{2\pi} \sum_{p=-\infty}^{\infty} \frac{x_p - x_q}{(x_p - x_q)^2 + (y_p - y_q)^2}$$

where $p=q$ is excluded from the summation. If now the vortices are disturbed from their equilibrium position, the small displacements being ξ_p, η_p , the vortex velocities can be developed in terms of these quantities, and we will be brought to a system of differential equations for the disturbances ξ_p, η_p , i. e., for small oscillations of the system.

Let us accordingly put

$$x_p = pl + \xi_p$$

$$y_p = \eta_p$$

and, neglecting the small quantities of higher orders, we will get

$$\frac{d\xi_q}{dt} = \frac{\Gamma}{2\pi} \sum_{p=-\infty}^{\infty} \frac{\eta_p - \eta_q}{(p-q)^2 l^2}$$

$$\frac{d\eta_q}{dt} = \frac{\Gamma}{2\pi} \sum_{p=-\infty}^{\infty} \frac{\xi_p - \xi_q}{(p-q)^2 l^2}$$

The differential equations so obtained, which are infinite in number, are reduced to two equations by the substitution

$$\xi_p = \xi_0 e^{ip\varphi}; \quad \eta_p = \eta_0 e^{ip\varphi}$$

These two equations are

$$\frac{d\xi_0}{dt} = \frac{\Gamma}{2\pi} \sum_{p=-\infty}^{\infty} \frac{e^{ip\varphi} - 1}{p^2 l^2}$$

$$\frac{d\eta_0}{dt} = \frac{\Gamma}{2\pi} \sum_{p=-\infty}^{\infty} \frac{e^{ip\varphi} - 1}{p^2 l^2}$$

with $p \neq 0$

The physical meaning of this substitution is easy to see: we consider a disturbance in which each vortex undergoes the same motion only with a different phase φ . Under such conditions we have to do with a wave disturbance and the system will be called stable, when for any value of φ , that is, for any phase difference between two consecutive vortices, the amplitude of the disturbance does not increase with the time.

Let us introduce the notation

$$\kappa(\varphi) = \frac{\Gamma}{2\pi} \sum_{p=-\infty}^{\infty} \frac{e^{ip\varphi} - 1}{p^2 l^2} = \frac{\Gamma}{\pi l^2} \sum_{p=1}^{\infty} \frac{\cos(p\varphi) - 1}{p^2}$$

The foregoing equations then take the form

$$\frac{d\xi_0}{dt} = \kappa(\varphi) \eta_0$$

$$\frac{d\eta_0}{dt} = \kappa(\varphi) \xi_0$$

Let us put ξ_0 and η_0 proportional to $e^{\lambda t}$; we will then find for each value of φ two values for λ , that is

$$\lambda = \pm \kappa(\varphi)$$

It follows that the vortex system considered is unstable for any periodic disturbance, because there is always present a positive real value of λ , that is, the disturbance is of increasing amplitude.

Applying this method in the case of two vortex rows we will find that the arrangement *a*, that is, the symmetrical arrangement, is likewise unstable, but that for the arrangement *b* there exists a value of the ratio h/l (h is the distance between the two rows, l is the distance between the vortices in the row) for which the system is stable.

In both cases λ can be brought to the form

$$\frac{\pi}{l}\lambda = \pm i(B \pm \sqrt{C^2 - A^2})$$

where A, B, C are functions of the phase difference φ . The system will be stable if $(C^2 - A^2)$ is positive for any value of φ .

For the symmetrical arrangement *a*, the functions A, B, C are expressed by the formulae:

$$A(\varphi) = \frac{1}{2h^2} \sum_{p=1}^{\infty} \frac{p^2 l^2 - h^2}{(p^2 l^2 + h^2)^3} + \sum_{p=1}^{\infty} \frac{1 - \cos p\varphi}{p^2 l^2}$$

$$B(\varphi) = \sum_{p=1}^{\infty} \frac{p^2 l^2 - h^2}{(p^2 l^2 + h^2)^3} \sin(p\varphi)$$

$$C(\varphi) = \frac{1}{2h^2} \sum_{p=1}^{\infty} \frac{p^2 l^2 - h^2}{(p^2 l^2 + h^2)^3} \cos(p\varphi)$$

But for $\varphi = \pi$ we get

$$A(\pi) = \frac{\pi^2}{8l^2} \left[\operatorname{ctgh}^2 \left(\frac{h\pi}{l} \right) - \operatorname{tgh}^2 \left(\frac{h\pi}{l} \right) \right]$$

$$C(\pi) = \frac{\pi^2}{8l^2} \left[\operatorname{ctgh}^2 \left(\frac{h\pi}{l} \right) - \operatorname{tgh}^2 \left(\frac{h\pi}{l} \right) \right]$$

so that this arrangement is unstable for any values of h and l .

For the unsymmetrical arrangement *b* we find

$$A(\varphi) = - \sum_{p=0}^{\infty} \frac{(p+\frac{1}{2})^2 l^2 - h^2}{[(p+\frac{1}{2})^2 l^2 + h^2]^3} + \sum_{p=1}^{\infty} \frac{1 - \cos(p\varphi)}{p^2 l^2}$$

$$B(\varphi) = \sum_{p=0}^{\infty} \frac{(p+\frac{1}{2})^2 l^2 - h^2}{[(p+\frac{1}{2})^2 l^2 + h^2]^3} \sin(p+\frac{1}{2})\varphi$$

$$C(\varphi) = \sum_{p=0}^{\infty} \frac{(p+\frac{1}{2})^2 l^2 - h^2}{[(p+\frac{1}{2})^2 l^2 + h^2]^3} \cos(p+\frac{1}{2})\varphi$$

We see now that $C(\pi) = 0$, so that in the place where $\varphi = \pi$, A must also be equal to zero, because, on account of the double sign, λ takes a positive real value. This brings us to the condition

$$\sum_{p=0}^{\infty} \frac{(p+\frac{1}{2})^2 l^2 - h^2}{[(p+\frac{1}{2})^2 l^2 + h^2]^3} = \sum_{p=0}^{\infty} \frac{2}{(2p+1)^2 l^2}$$

But

$$\sum_{p=0}^{\infty} \frac{[(p+\frac{1}{2})^2 l^2 - h^2]}{[(p+\frac{1}{2})^2 l^2 + h^2]^3} = \frac{\pi^2}{2l^2 \cosh^2 \frac{\pi h}{l}}$$

and

$$\sum_{p=0}^{\infty} \frac{2}{(2p+1)^2 l^2} = \frac{\pi^2}{4l^2}$$

so that, as the necessary condition of stability we find the relation

$$\cosh \frac{h\pi}{l} = \sqrt{2}$$

and for the ratio h/l we find the value

$$h/l = 0.283 \dots$$

For a certain value of the wave length of the disturbance, corresponding to $\varphi=\pi$, we get $\lambda=0$, that is, the system is in a neutral state. But it can be shown by calculation that our system is stable for all other disturbances. This unique disturbance has to be tested by further investigations. It can, however, be seen that a zero value for λ must appear, because only one stable configuration exists. If this were not so, we would find for l/h a finite domain of stability.¹

THE "FLOW PICTURE."

The consideration of the question of stability has brought us to the result that there exists a particular configuration of two vortex rows which is stable. The vortices of both rows have then such an arrangement that the vortices of one row are placed opposite the middle of the interval between the vortices of the other row, and the ratio of the distance h between the two rows to the distance l between the vortices of the same row has the value

$$\frac{h}{l} = \frac{1}{\pi} \operatorname{arc} \cosh \sqrt{2} = 0,283$$

The whole system has the velocity

$$u = \frac{\zeta}{\pi} \sum_{p=0}^{\infty} \frac{h}{(p+\frac{1}{2})^2 l^2 + h^2}$$

which can also be written

$$u = \frac{\zeta}{2l} \operatorname{tgh} \frac{\pi h}{l}$$

or, introducing the value of h/l found by the stability investigation, we get

$$u = \frac{\zeta}{l\sqrt{8}}$$

The flow is given by the complex potential (φ potential, ψ flow function)

$$z = \varphi + i\psi = \frac{i\zeta}{2\pi} \lg \frac{\sin(z_0 - z)\frac{\pi}{l}}{\sin(z_0 + z)\frac{\pi}{l}}$$

where

$$z_0 = \frac{l}{4} + \frac{hl}{2}$$

By aid of this formula we have calculated the corresponding streamlines and have represented them in Fig. 2. We see that some of the streamlines are closed curves around the vortices, while the others run between the vortices. On the other hand, we have tried to make visible the flow picture behind a body, e. g., a flat plate or circular cylinder, moved through immobile water, by aid of lycopodium powder sifted on the surface of the water, and to fix these pictures photographically (exposure one-tenth of a second).

The regularly alternated arrangement of the vortices can not be doubted. In most cases the vortex centers can also be well determined; sometimes the picture is disturbed by small "accidental vortices" produced in all probability by small vibrations of the body, which in our provisional experiments could not be avoided. We had a narrow tank whose floor was formed by a band running on two rolls, and the bodies tested were simply put on the moving band and carried by it. It is to be expected

that by aid of an arrangement especially made for the purpose much more regular flow pictures could be obtained, while in the actual experiments the flow was disturbed on the one hand by the vibrations of the body and on the other by the water flow produced by the moving band itself.

The alternated arrangement of the vortices rotating to the right and to the left can only be obtained when the vortices periodically run off first from one side of the body, then from the other, and so on, so that behind the body there appears a periodic motion, oscillating from one side to the other, but with such a regularity, however, that the frequency of this oscillation can be estimated with sufficient exactness. The periodic character of the motion in the so-called "vortex wake" has often been observed. Thus, Bernard² has remarked that the flow picture behind a narrow obstacle can be decomposed into vortex fields with alternated rotations. Also for the flow of water around balloon models the oscillation of the vortex field has been observed.³ Finally, v. d. Borne⁴ has observed and photographed recently the alternated formation of vortices in the case of air flowing around different obstacles. The

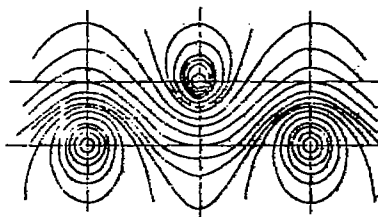


FIG. 2.

¹ From a mathematical standpoint our stability investigation may be considered as a direct application of the theorems of Mr. O. Toplit on Cyclanten with an infinite number of elements, which he has in part published in two papers (Gottingen Nachrichten, 1907, p. 110; Math. Annalen 1911, p. 351), and in part been so kind as to communicate personally to us.

² Comptes Rendus, Paris, 143, 839, 1908.

³ Technical report of the Advisory Committee for Aeronautics (British), 1910-11.

⁴ Undertaken on the initiative of the representatives of aeronautical science in Gottingen, November, 1911.

phenomenon could not be explained until now; according to our stability investigation the periodic variations appear as a natural consequence of the instability of the symmetrical flow.¹

It is also very interesting to observe how the stable configuration is established. When, for example, a body is set in motion from rest (or conversely, the stream is directed onto the body) some kind of "separation layer" is first formed, which gradually rolls itself up, at first symmetrically on both sides of the body, till some small disturbance destroys the symmetry, after which the periodic motion starts. The oscillatory motion is then maintained corresponding to the regular formation of left hand and right hand vortices.

We have also made a second series of photographs for the case of a body placed at rest in a uniform stream of water. For this case the flow picture can be obtained from Fig. 2 by the superposition of a uniform horizontal velocity. We will then see on the lines drawn through the vortex centers perpendicular to the stream direction, some ebbing point where the stream lines intersect and the velocity is equal to zero. However, in the same way as the motion is affected by the vibrations of the experimental body in the case of the motion of a body in the fluid, so in this case the turbulence of the water stream gives rise to disturbances.

As to the quantitative agreement attained by the theory, it must be noted that our stability conditions refer to infinite vortex rows, so that an agreement of the ratio h/l with the measured values is to be expected only at a certain distance from the body. The measurements on the photographs show that the distance l between vortices in a row is very regular, so that l may be measured satisfactorily, but per contra the distance h is much more variable, because the disturbance of the vortices takes place principally in the direction normal to the rows, that is, the latter undergo in the main transverse oscillations. The best way to determine the mean positions of the centers of the vortices would be by aid of cinematography, but we can also, without any special difficulty, find by comparison the mean direction of each vortex row directly from photographs. So in the case of the photograph of a circular cylinder 1.5 cm. in diameter, when making measurements beyond the first two or three vortex pairs we have found the following mean values for h and l

$$h=1.8 \text{ cm.}; l=6.4 \text{ cm.}$$

So that for the ratio h/l we obtain the value

$$h/l=0.28.$$

For the flow around a plate of 1.75 cm. breadth we found

$$h=3 \text{ cm.}; l=9.8 \text{ cm.}$$

Accordingly

$$h/l=0.305.$$

The agreement with the theoretical value 0.283 is entirely satisfactory.

For the first vortex pair behind the body, h/l comes out sensibly larger, somewhere near $h/l=0.35$. But in the first investigation of Kármán, mentioned at the beginning of this paper, the stability of the vortex system was investigated in such a way that all the vortices with the exception of one pair were maintained at rest and the free vortex pair considered oscillating in the velocity field of the others. Under such assumptions it was found that $h/l=1/\pi \operatorname{arc} \cosh \sqrt{3}=0.36$. We therefore think that the conclusion can be drawn, that in the neighborhood of the body, where the vortices are even more limited in their displacements, the ratio h/l is greater than 0.283 and approaches rather the value of 0.36.

APPLICATION OF THE MOMENTUM THEOREM TO THE CALCULATION OF FLUID RESISTANCE.

Let us assume that at a certain distance behind the body there exists a flow differing but slightly from the one of stable configuration which we have established theoretically in the foregoing, but that at a distance in front of the body, which is great in comparison with the size of the body, the fluid is at rest—as it is quite natural to assume. We will then be brought by the application of the momentum theorem to a quite definite expression for the resistance which a body moving with a uniform velocity in a fluid must experience. Practically, by such a calculation for the uniplanar problem, we will obtain the resistance of a unit of length of an infinite body placed normally to the plane of the flow.

We will use a system of coordinates moving with the same speed u as the vortex system behind the body. In this coordinate system, according to our assumptions, at a sufficient distance from the body the vortex motion behind the body as well as the fluid state in front of the body will be steady, and we will have, when referred to this system of coordinates, a uniform flow of speed $-u$ in front of the body, but behind the body the velocity components will be expressed by

$$-u + \frac{\partial \psi}{\partial y} \text{ and } -\frac{\partial \psi}{\partial x}$$

where ψ is the real part of the complex potential

$$x = \varphi + i\psi = \frac{it}{2\pi} \lg \frac{\sin(z_0 + s)}{\sin(z_0 - s)}$$

¹ The tone that is emitted by a stick rapidly displaced in air is fixed by this periodicity, to which Prof. C. Runge has already drawn our attention.

The body itself has, relative to this system of coordinates, the velocity $U-u$, where U is the absolute velocity of the body. If we designate by l the distance between the vortices of one row, there must take place, as a consequence of the displacement of the body, in the time $T=l/(U-u)$, the formation of a vortex on each side of the body. We will calculate the increment of the momentum, along the X axis, in this time interval T (that is, between two instants of time of identical flow state) and for a part of the flow plane, which we define in the following way (see fig. 3). On the sides the plane portion considered is limited by the two parallel straight lines $y=\pm\eta$; in front and behind, by two straight lines $x=\text{Const.}$ disposed at distances from the body which are great in comparison with the size of the body, the line behind the body being drawn so as to pass through the point half way between two vortices having inverse rotation. When the boundary lines are sufficiently far from the body we can consider the fluid velocities at those lines as having the values indicated in the foregoing.

For a space with the boundaries indicated above the relation must exist that the momentum imparted to the body $\int_0^T W dt$ (where W is the resultant fluid resistance) is equal to the difference between the momentum contained in the space considered at the times $t=\tau$ and $t=\tau+T$ and, the sum of the inflow momentum and the time integral of the pressure along the boundary lines. If we thus consider as exterior forces the force $-W$ and the pressure, which act on the whole system of fluid and solid, they must then correspond to the increment of the momentum—that is, to the excess of momentum after the time T less the inflow momentum.

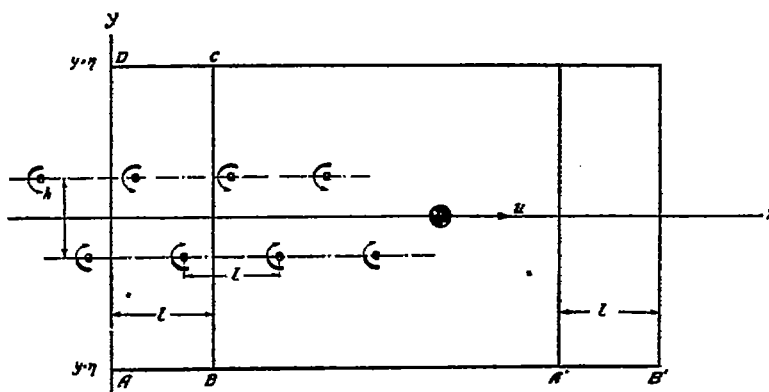


FIG. 3.

We will calculate these momentum parts separately. The excess of momentum after the time T is equal to the difference of the values that the double integral $\rho \iint u(x, y) dx dy$ takes at the times $t=\tau$ and $t=\tau+T$. But the time interval has been chosen in such a way that the state of flow is identical, with the difference that the body has been displaced through the distance $l=(U-u)T$. The double integral reduces thus to the difference of the integrals taken over the strips $ABCD$ and $A'B'C'D'$ both of breadth l . For the strip $A'B'C'D'$ the fluid speed can be taken equal to $-u$ for the strip $ABCD$ equal to $-u + \frac{\partial \psi}{\partial y}$ so that we get

$$I_1 = \rho \int_0^l \int_{-\eta}^{\eta} \frac{\partial \psi}{\partial y} dx dy$$

If we pass to side boundaries having $\eta=\infty$, we obtain for I_1 the very simple expression

$$I_1 = \rho \dot{\Gamma} h$$

which can also be obtained directly by the application of the general momentum theorem to vortex systems.

We will unite in one single term the inflow momentum and the time integral of the pressure, because in such a way we will be led to more simple results. If we consider a uniplanar steady fluid motion with the velocity components $u(x, y)$ and $v(x, y)$ and consider a fixed contour in the plane, the inflow momentum in a unit of time in the direction of X is expressed by the closed integral $\rho \int (\bar{u}^2 dy - \bar{u} \bar{v} dx)$ where \bar{u} , \bar{v} are the velocities on the contour. The pressure gives the resultant $\int \bar{p} dy$ along the X axis, but since for a steady flow the relation

$$p = \text{Const} - \rho \frac{\bar{u}^2 + \bar{v}^2}{2}$$

must hold, we thus obtain for the sum of both integrals, multiplied by T

$$\begin{aligned} I_2 &= T \int \rho (\bar{u}^2 dy - \bar{u} \bar{v} dx) + T \int \bar{p} dy \\ &= T \rho \int \left(\frac{\bar{u}^2 - \bar{v}^2}{2} dy - \bar{u} \bar{v} dx \right) \end{aligned}$$

Or, introducing the complex quantity,

$$\bar{w} = \bar{u} - i\bar{v} = \frac{\partial(\varphi + i\psi)}{\partial(x + iy)} = \frac{\partial\chi}{\partial z}$$

we get

$$I_2 = \rho \operatorname{Im} \int (\bar{w}^2 dz)$$

where Im is to be understood as the complex part of the integral.

If we put for the contour

$$\begin{aligned}\bar{u} &= -u + u' \\ \bar{v} &= v'\end{aligned}$$

then the terms in u^2 will at once be eliminated, and also the terms in u on account of the equality of the inflow and outflow; and there will remain only the terms in u'^2 and $u'v'$. The latter will give a finite value only for the boundary line passing through the vortex system (AD in fig. 3). Passing to $\eta = \infty$, we get

$$I_2 = T\rho \operatorname{Im} \left[\int_{-i\infty}^{i\infty} \left(\frac{d\chi}{dz} \right)^2 dz \right]$$

and integrating along AD we get

$$I_2 = T\rho \operatorname{Im} \left[\int_{\chi(-i\infty)}^{\chi(i\infty)} \frac{d\chi}{dz} d\chi \right]$$

But

$$w = \frac{d\chi}{dz} = -\frac{\xi}{l} \operatorname{tgh} \frac{h\pi}{l} - \frac{i\xi \cos \frac{2\pi x}{l}}{l \cosh \frac{h\pi}{l}}$$

so that, integrating and introducing the values

$$\chi(iw) = \frac{\xi}{4} - i\frac{h\xi}{2l}$$

$$\chi(-iw) = -\frac{\xi}{4} + i\frac{h\xi}{2l}$$

$$I_2 = T\rho \left[\frac{\xi u h}{l} - \frac{\xi^2}{2\pi l} \right]$$

where u again has been written for $\frac{\xi}{2l} \operatorname{tgh} \frac{\pi h}{l}$.

Thus the total momentum imparted to the body is

$$\int_0^T W dt = \rho \xi h - T\rho \left(\frac{\xi u h}{l} - \frac{\xi^2}{2\pi l} \right)$$

If for the mean value of $1/T \int_0^T W dt$ we write \bar{W} (as the time mean value of the resistance) we will obtain with $T = l/(U - u)$ the final formula:

$$(II) \quad \bar{W} = \rho \xi \frac{h}{l} (U - 2u) + \rho \frac{\xi^2}{2\pi l}$$

The fluid resistance appears here expressed by the three characteristic constants ξ , h , l of the vortex configuration (as u is expressed by the last). In the deduction of this last formula we did not take account of the stability conditions, so that this formula applies to any value of the ratio h/l . If we assume the vortices in the row to be brought all close together so that they are uniformly distributed along the row, but in such a way that the vortex intensity per unit of length remains finite, we thus pass to the case of continuous vortex sheets. In this case $\xi/l = U$, but $\xi^2/l = 0$ and $u = \frac{U}{2}$, so that the fluid resistance disappears. The discontinuous potential flow of v. Helmholtz thus does not give any resistance when the depth of the dead water remains finite, as can also be shown from general theorems.

THE FORMULAE FOR FLUID RESISTANCE.

Let us now apply to our special case the general formula we have just found, introducing the relations between ξ and u , and h and l according to the stability conditions. For the speed u we have

$$u = \frac{\xi}{\sqrt{8}}$$

further,

$$h/l=0,283$$

so that we get

$$W=\rho l \left[0,283\sqrt{8}u (U-2u) + \frac{4}{\pi}u^2 \right]$$

If we introduce, as is ordinarily done, the resistance coefficient according to the formula

$$W=\psi_w \rho d U^2$$

where d is a chosen characteristic dimension of the body, to which we refer the resistance, we will obtain ψ_w expressed by the two ratios u/U and l/d in the following way

$$(III) \quad \psi_w = \left[0,799 \frac{u}{U} - 0,323 \left(\frac{u}{U} \right)^2 \right] \frac{l}{d}$$

We have thus obtained the resistance coefficient—which before could be observed only by resistance measurements—expressed by two quantities which can be taken directly from the flow phenomenon, viz, the ratio

$$\frac{u}{U} = \frac{\text{Velocity of the vortex system}}{\text{Velocity of the body}}$$

and

$$\frac{l}{d} = \frac{\text{Distances apart of the vortices in one row}}{\text{Reference dimension of the body}}$$

Both quantities, corresponding to the similitude of the phenomenon, within the limits of validity of the square law can depend only upon the dimension of the body.

These two quantities can be observed very easily experimentally. The ratio l/d can be taken directly from photographs, while the ratio u/U can be found easily by counting the number of vortices formed. If we designate by T the time between two identical flow states we can then introduce the quantity $l_0 = UT$, which is the distance the body moves in the period T . This quantity must be independent of velocity for the same body, and the ratio l/l_0 for similar bodies must also be independent of the dimensions of the body but determined by the shape of the body. Remembering that $T=l/(U-u)$, we then find between u/U and l/l_0 the simple relation

$$\frac{u}{U} = 1 - \frac{l}{l_0}$$

By some provisional measurements we have proved the similitude rule and afterwards calculated the resistance coefficient for a flat plate and a cylinder disposed normal to the stream, for the purpose of seeing if the calculated values agreed with the air resistance measurements, at least in order of magnitude.

Our measurements were made first on two plates of width 1.75 and 2.70 cm. and 25 cm. length, and we have measured the period T and calculated the quantity $l_0 = UT$ for two different velocities. We have used a chronograph for time measurements and the period was observed for each vortex row independently. Thus was found for the narrower plate

$$\begin{array}{ll} U=10.0 \text{ cm/sec} & 15.1 \text{ cm/sec} \\ T=1.28 \text{ sec} & 0.805 \text{ sec} \\ UT=12.6 \text{ cm} & 12.1 \text{ cm} \end{array}$$

for the wider plate

$$\begin{array}{ll} U=9.6 \text{ cm/sec} & 15.5 \text{ cm/sec} \\ T=1.99 \text{ sec.} & 1.20 \text{ sec.} \\ UT=19.1 \text{ sec.} & 18.6 \text{ sec.} \\ \text{Mean value } UT=18.8 \text{ cm} \end{array}$$

The ratio of the plate width is equal to

$$\frac{2.70}{1.75} = 1.54$$

and the ratio of the quantities $l_0 = UT$ is equal to

$$\frac{18.8}{12.3} = 1.52$$

So that the similitude rule is in any case confirmed.

A circular cylinder of 1.5 cm. diameter was also tested at two speeds. We found the values

$$\begin{array}{ll} U=11.0 \text{ cm/sec} & 15.8 \text{ cm/sec} \\ T=0.66 \text{ sec.} & 0.48 \text{ sec.} \\ UT=7.3 \text{ cm} & 7.5 \text{ cm} \end{array}$$

$$\text{Mean value } UT=7.4 \text{ cm}$$

Knowing the values of $l_0 = UT$ we can calculate for the plate and the cylinder the speed ratio u/U . Thus,

for the plate $u/U = 0.20$.
for the cylinder $u/U = 0.14$

and with the values of l indicated before we have

for the plate $l/d = 5.5$
for the cylinder $l/d = 4.3$

where d is the plate width or cylinder diameter. We thus find the resistance coefficients

for the plate $\psi_w = 0.80$
for the cylinder $\psi_w = 0.46$

The resistance measurements of Foppl¹ have given for a plate with an aspect ratio of 10:1 the resistance coefficient $\psi_w = 0.72$ and the Eiffel² measurements, for an aspect ratio of 50:1; that is, for a nearly plane flow, the value $\psi_w = 0.78$. Further, Foppl has found for a long circular cylinder $\psi_w = 0.45$, so that the agreement between the calculated and measured resistance coefficients must be considered as fully satisfactory.

The theoretical investigations here developed ought to be extended and completed in two directions. First, we have limited ourselves to the uniplanar problem; that is, to the limiting case of a body of great length in the direction normal to the flow. It is to be expected that by the investigation of stable vortex configurations in space we will also be brought to a better understanding of the mechanism of fluid resistance. However, the problem is rendered difficult by the fact that the translation velocity of curved vortex filaments is not any longer independent of the size of the vortex section, because to an infinitely thin filament would correspond an infinitely great velocity. Nevertheless, it must not be considered that the extension of the theory to the case of space would bring unsurmountable difficulties.

Much more difficult appears the extension of the theory in another direction, which really would first lead to a complete understanding of the theory of fluid resistance, namely, the evaluation by pure calculation of the ratios l/d and u/U , which we have found from flow observations, and which determine the fluid resistance. This problem can not be solved without investigation of the process of vortex formation. An apparent contradiction is brought out by the fact that we have used only the theorems established for perfect fluids, which in such a fluid (frictionless fluid) no vortices can be formed. This contradiction is explained by the fact that we can everywhere neglect friction except at the surface of the body. It can be shown that the friction forces tend to zero when the friction coefficient decreases, but the vortex intensity remains finite. If we thus consider the perfect fluid as the limiting case of a viscous fluid, then the law of vortex formation must be limited by the condition that only those fluid particles can receive rotation which have been in contact with the surface of the body.

This idea appears first, in a perfectly clear way, in the Prandtl theory of fluids having small friction. The Prandtl theory investigates those phenomena which take place in a layer at the surface of the body, and the way in which the separation of the flow from the surface of the body occurs. If we could succeed in bringing into relation these investigations on the method of separation of the stream from the wall with the calculation of stable configuration of vortex films formed in any way whatever, as has been explained in the foregoing pages, then this would evidently mean great progress. Whether or not this would meet with great difficulties can not at the present time be stated.

¹ See the work of O. Foppl already mentioned.

² G. Eiffel, "La Resistance de l'Air et l'Aviation," p. 47, Paris, 1910.